VUV and X-ray Free-Electron Lasers

SASE and High-Gain FEL Theory

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Wednesday Schedule

• Self-Amplified Spontaneous Emission	09:00 – 10:00
• Break	10:00 – 10:10
• 1D Theory of High-Gain FEL	10:10 – 11:10
• Break	11:10 – 11:20
• Ming-Xie Parameterization of 3D Effects	11:20 – 12:00
• Lunch Break	12:00 – 13:30
• Lab Project	13:30 – 17:00

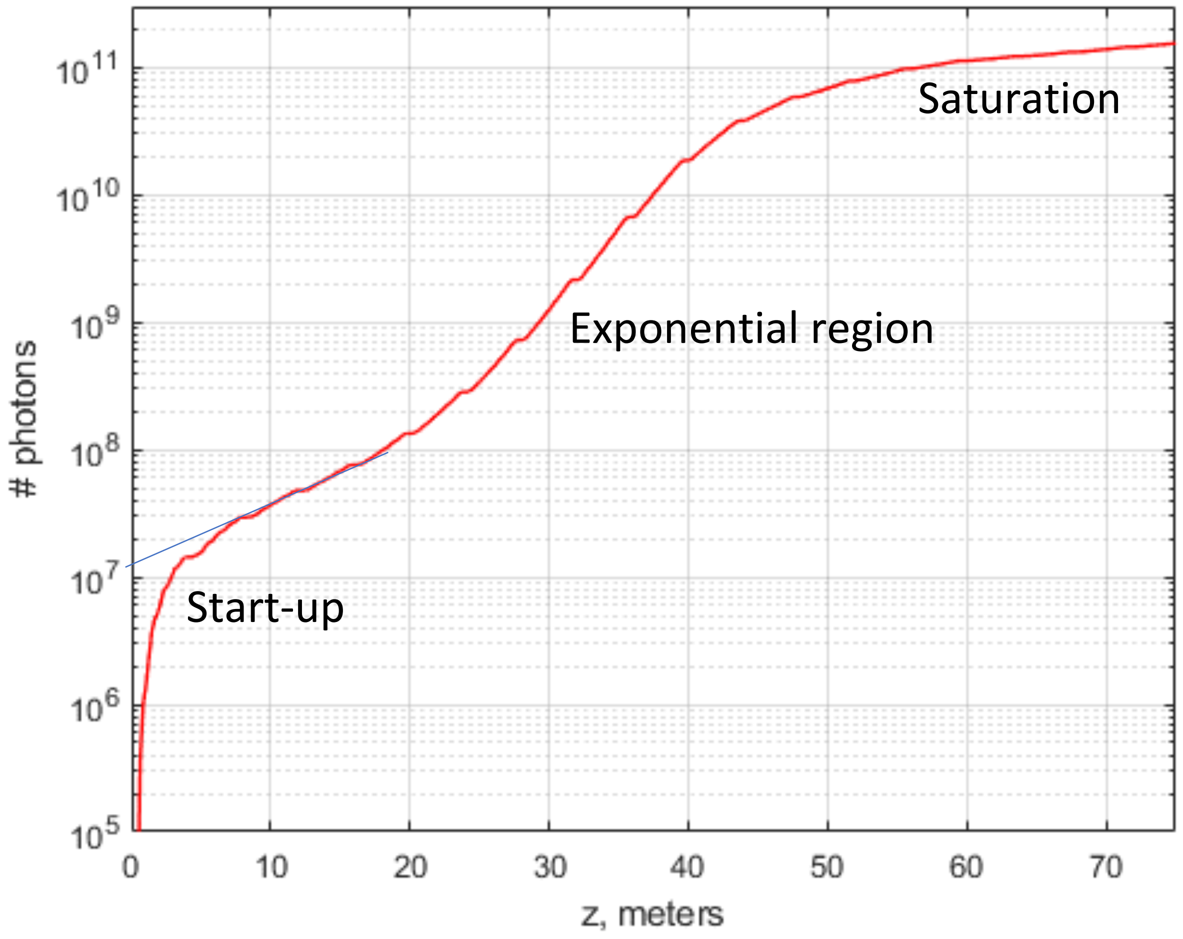
Self-Amplified Spontaneous Emission

SASE is most common approach in XFELs

- Start with high-brightness electron beams.
  - Low emittance
  - High peak current
  - Low energy spread
- Match the electron beams into a long undulator with a quad FODO lattice.
- Produce high FEL gains.

$$\rho = \frac{1}{\gamma_r}\left(\frac{\hat{K}\,\lambda_u}{8\pi\sigma}\right)^{\frac{2}{3}}\left(\frac{I_p}{I_A}\right)^{\frac{1}{3}}$$

- Optimize the FEL to shorten the gain length.
- FEL power saturates after 20 gain lengths.



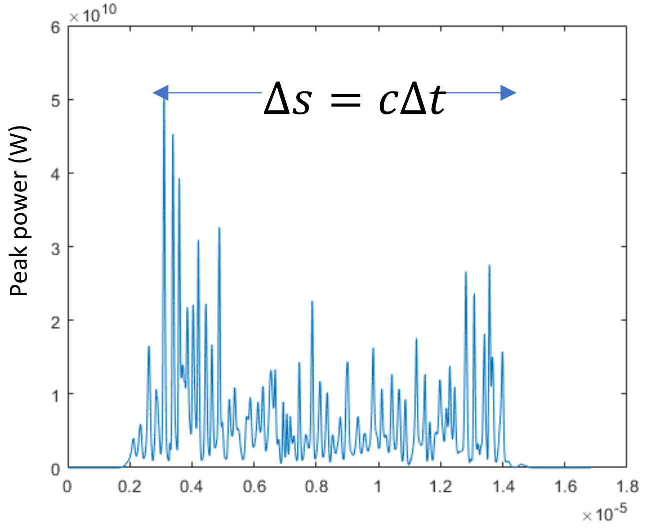
SASE Characteristics

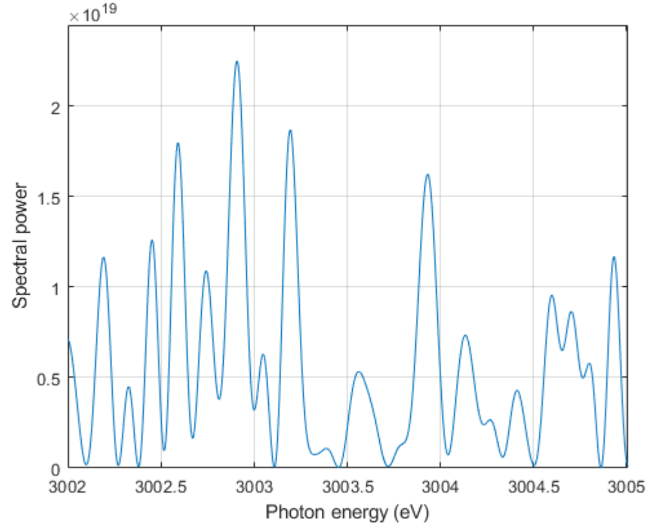
- Radiation starts up from “**white noise**” radiation due to the discrete nature of electrons.
- The FEL interaction amplifies the “white noise” within a **narrow FEL gain spectrum**.
- In a long undulator with strong focusing, the FEL enters the **exponential growth** regime with a characteristic gain length.
- **FEL instability**:

  High field region → Large energy modulation

  Large energy modulation → Strong bunching

  Strong bunching → Higher field
- The randomness of the initial bunching is apparent in the final temporal profile and SASE spectrum with both exhibiting “**spikes**.”





Start-up Noise

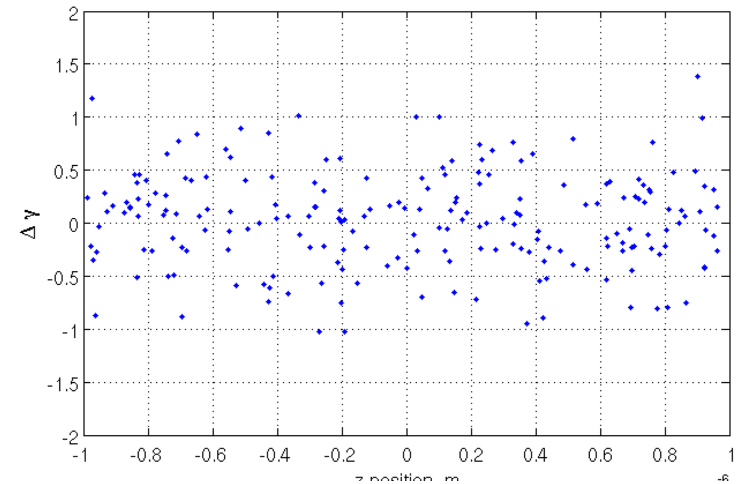
We have been describing electron current as a smooth function $I(t)$. A more accurate description of current, which accounts for the discrete nature of electrons, is a sum of Dirac delta functions:

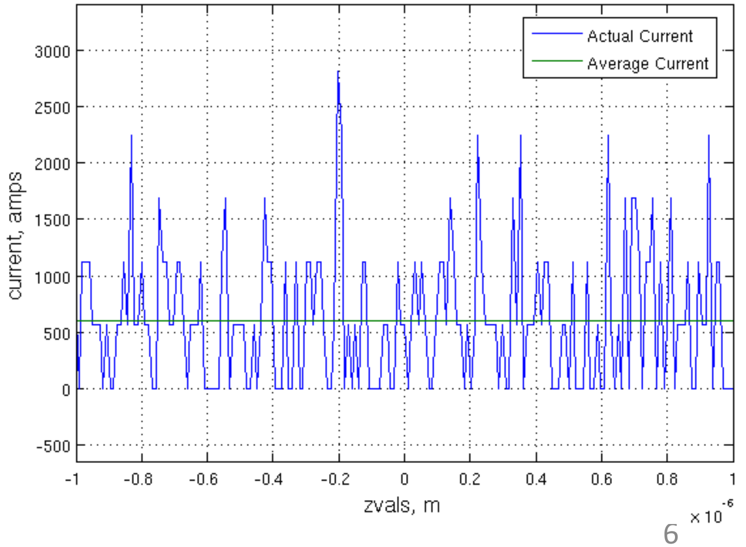
$$I(t) = e\sum_{j=1}^{N} \delta(t - t_j)$$

Taking the Fourier Transform of $I(t)$:

$$i_T(\omega) = \int_{-\infty}^{+\infty} \left[e\sum_{j=1}^{N} \delta(t - t_j) \right] \exp(i\omega t)\, dt = e\sum_{j=1}^{N} \exp(i\omega t_j)$$

$$S(\omega) = \frac{1}{\pi T} \langle |i_T(\omega)|^2 \rangle$$





SASE Random Behavior

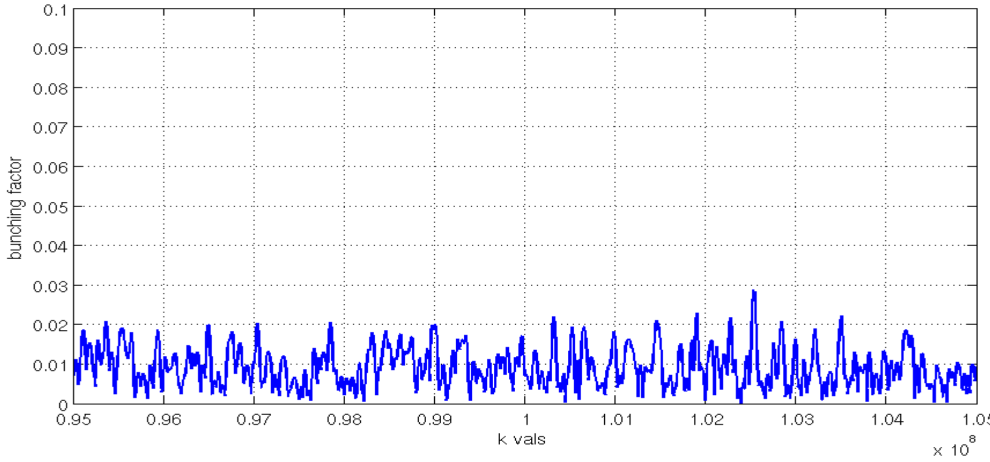
- The discrete nature of the electrons leads to random fluctuations in current as a function of s.

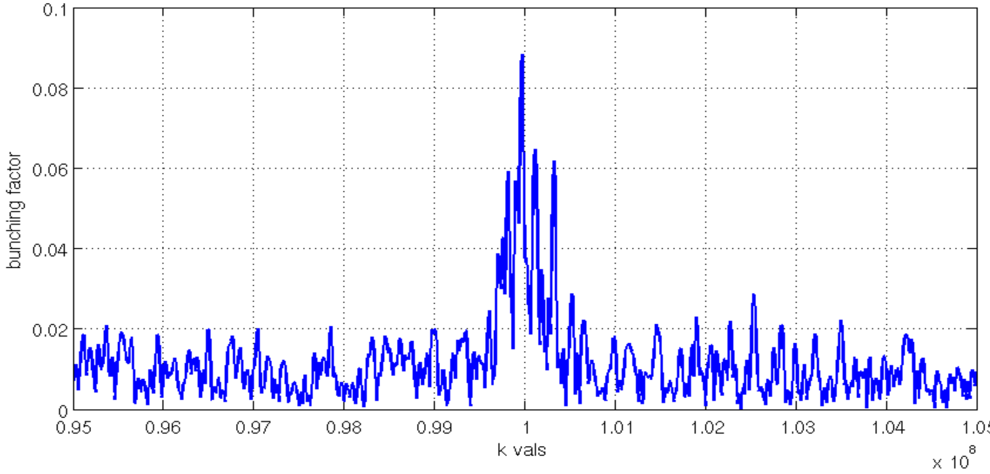
- Taking the Fourier transform of current fluctuations yields "white noise" in the frequency domain, i.e. the bunching factor versus frequency is random.

- The FEL amplifies a narrow portion of the "white noise" spectrum. This portion of the spectrum grows to high power. The randomness of the initial bunching is still apparent in the final SASE spectrum.

Gain Bandwidth for $z \geq 8L_G$ **before saturation**

$$\left(\frac{\sigma_\omega}{\omega}\right)_z = 3\sqrt{2}\,\rho\sqrt{\frac{L_G}{z}}$$





Start-up power scaling

- The spontaneous noise power and the number of photons at startup are given by

$$P_{su} = \frac{3}{2}\frac{\rho^2 P_b}{N_\lambda}$$

Electrons beam power

# of electrons/wavelength

- As the FEL wavelength decreases (higher photon energy), the # of electrons/wavelength decreases and the start-up power increases.

Wavelength Region	Estimated Start-up Power (W)
Visible	1
EUV	10
Soft X-rays	100
Hard X-rays	1000

Power Growth Curves

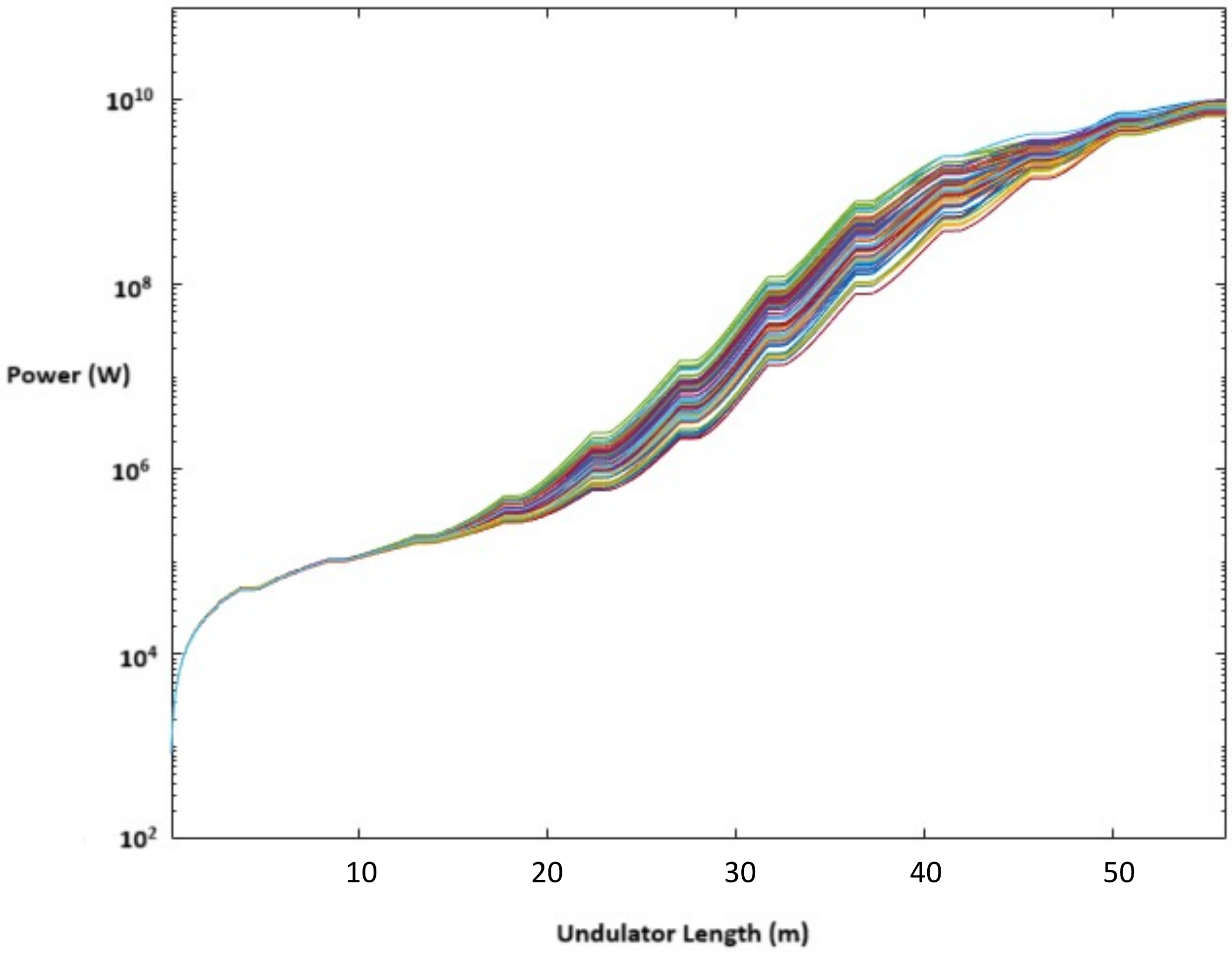
In the exponential region, the slope of the semi-log plot (for one e-folding) is equal to

$$Slope \propto \frac{1}{L_G}$$

The peak power fluctuates wildly in the exponential growth regime due to the stochastic nature of the FEL instability.

At saturation, the pulse-to-pulse amplitude fluctuations decrease and FEL power can be approximated by

$$P_{sat} \approx \frac{\rho E_b I_p}{e}$$

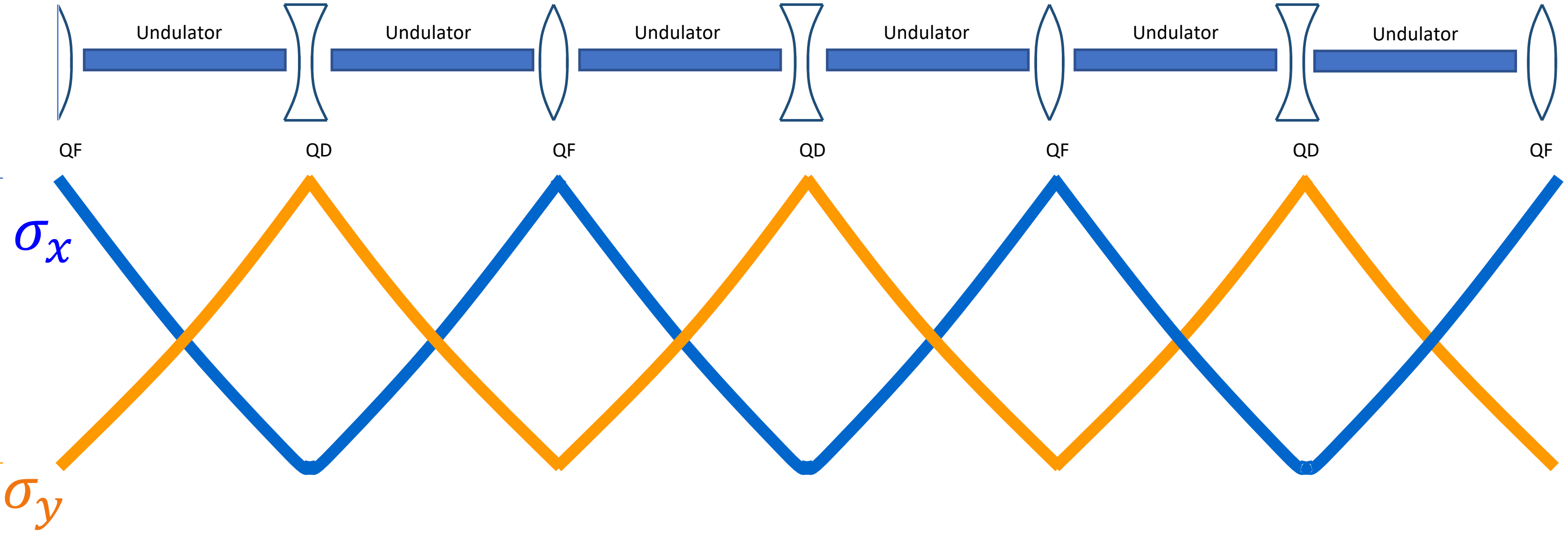


Strong Focusing in an Undulator FODO

In a FODO lattice, the electron beam radii in x and y oscillate between a maximum and minimum values set by the β functions and the un-normalized emittance in x and y. We consider the case where $\varepsilon_x = \varepsilon_y$ and $\beta_x > \beta_y$

$$\sigma_{max} = \sqrt{\beta_x \varepsilon_x}$$

$$\sigma_{min} = \sqrt{\beta_y \varepsilon_y}$$



Electron beam rms angle

$$\sigma_{x'} = \sqrt{\varepsilon_x \frac{(1+\alpha_x^2)}{\beta_x}} = \sigma_{y'} = \sqrt{\varepsilon_y \frac{(1+\alpha_y^2)}{\beta_y}}$$

Optical Diffraction

The rms radius of the radiation beam is determined by two competing effects: optical guiding (beam focusing) and diffraction (beam expanding). The minimum radiation radius is approximately the larger of the x and y electron beam radii in the FODO lattice.

$$\sigma_r \geq \max(\sigma_{x,y})$$

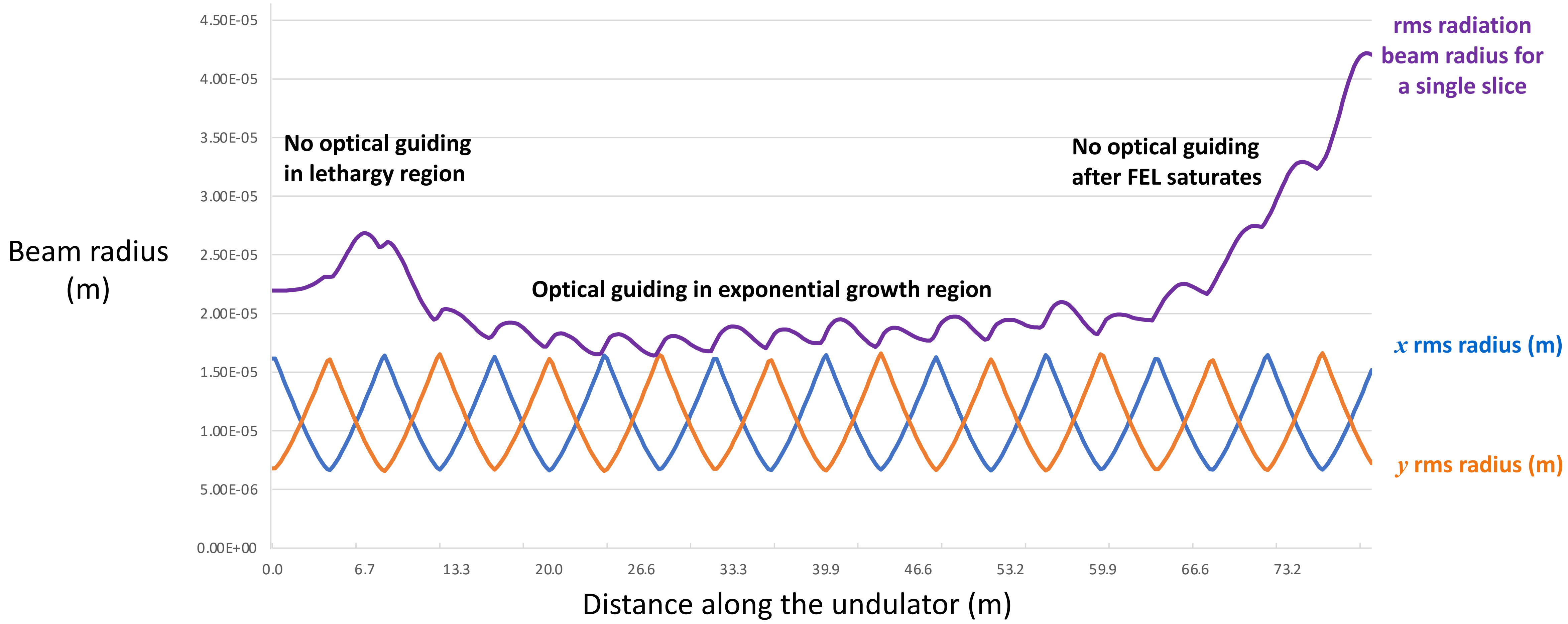
The Rayleigh length is chosen to minimize the effect of optical diffraction.

$$\sigma_{r'} = \frac{\sigma_r}{z_R} \qquad z_R = \frac{4\pi\sigma_r^2}{\lambda}$$

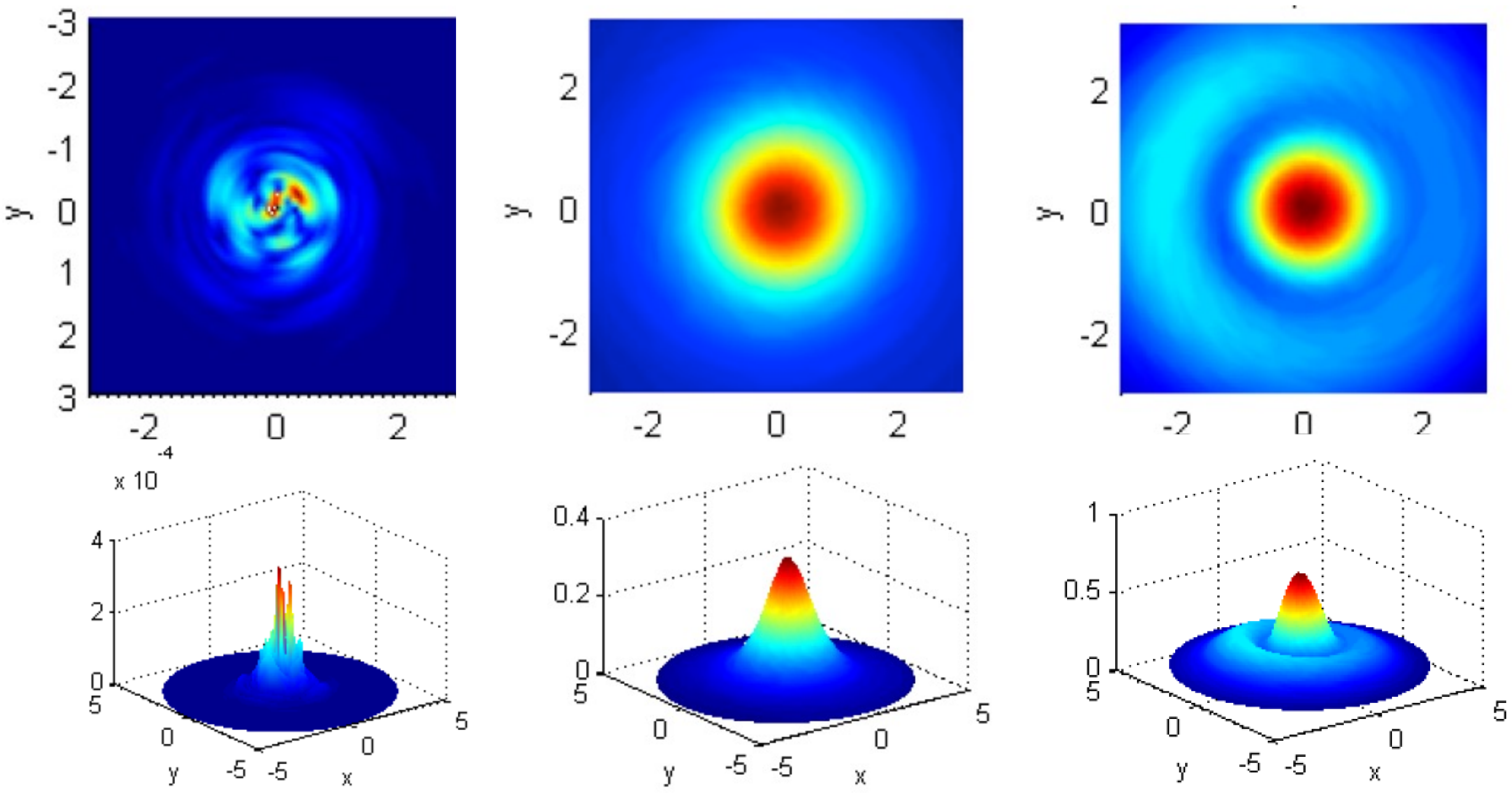
To minimize diffraction effect, the radiation Rayleigh range must be longer than the 1D gain length

$$z_R > L_{1D}$$

Optical Guiding in an Undulator FODO

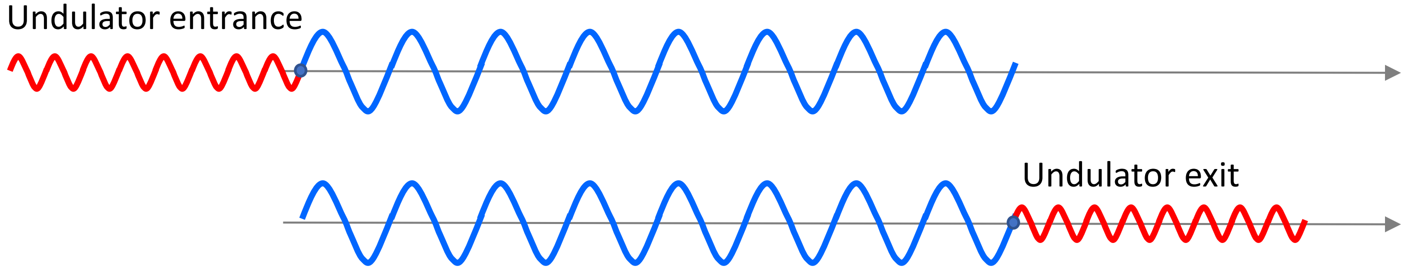


SASE Transverse Coherence



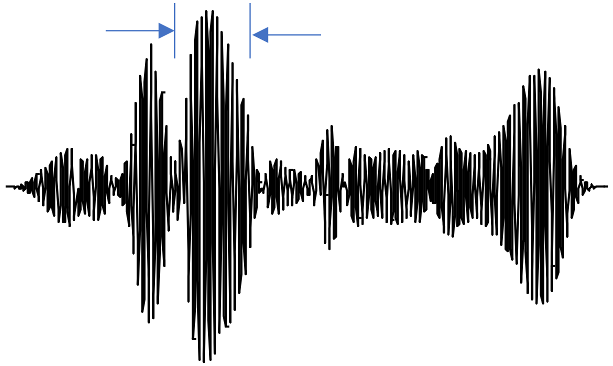
Incoherent Spontaneous Emission | SASE in exponential region | SASE at saturation

Slippage Length & Coherence Length



$$l_{slippage} = N_u \lambda$$

Coherence length, l_c



In the time the electron traverses N_u undulator periods, the optical wave slips ahead of the electron N_u wavelengths, a distance known as the slippage length. The coherence length is the slippage length over one gain length.

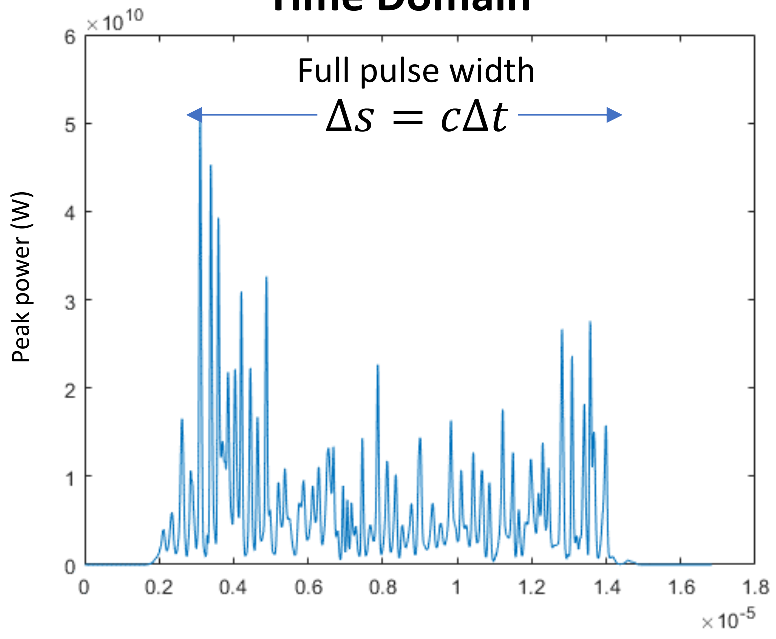
$$l_c = \frac{\lambda}{4\sqrt{\pi}\rho}$$

In a SASE FEL, the radiation coherence extends over only one coherence length. For bunch length longer than the coherence length, each coherence length is independent from the others.

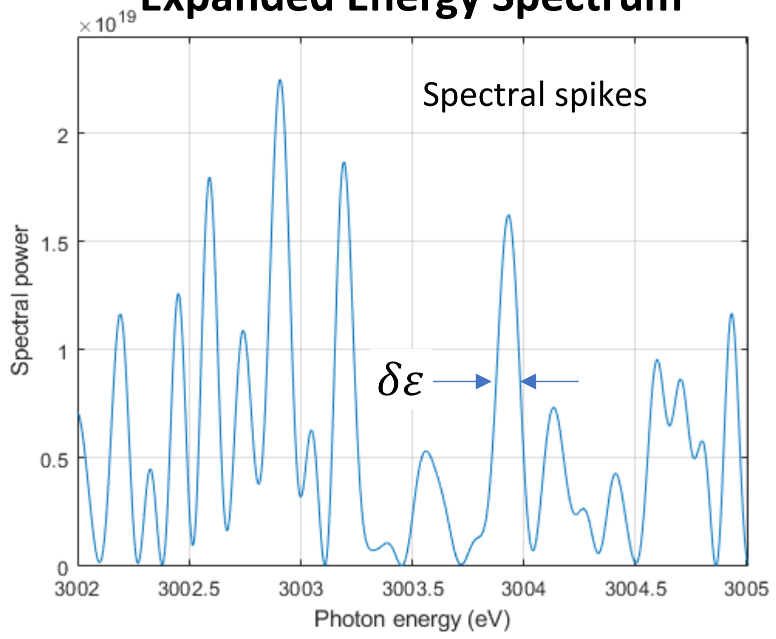
The number of coherence length in a SASE pulse $N_c \approx \frac{c\Delta t_b}{l_c}$

SASE Time-Bandwidth

Time Domain



Expanded Energy Spectrum

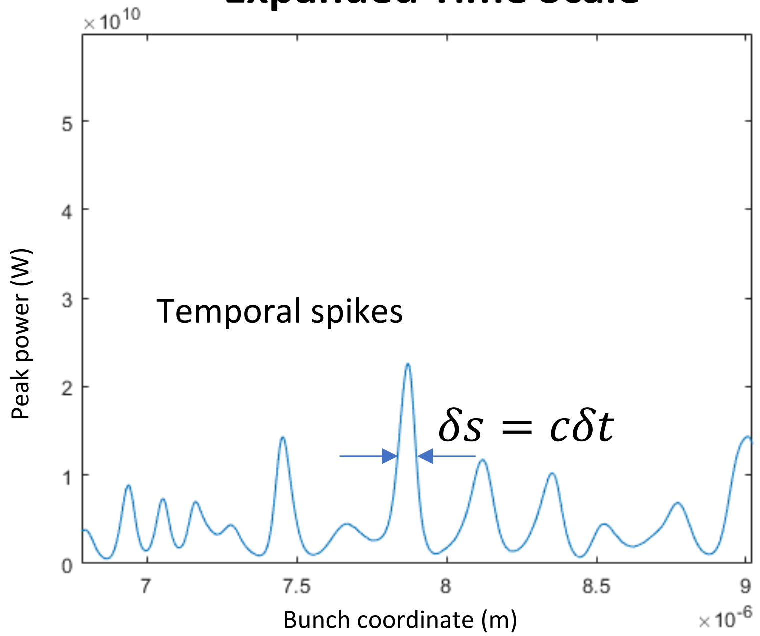


$$\delta\varepsilon \cdot \Delta t = 1.8\ eV \cdot fs$$

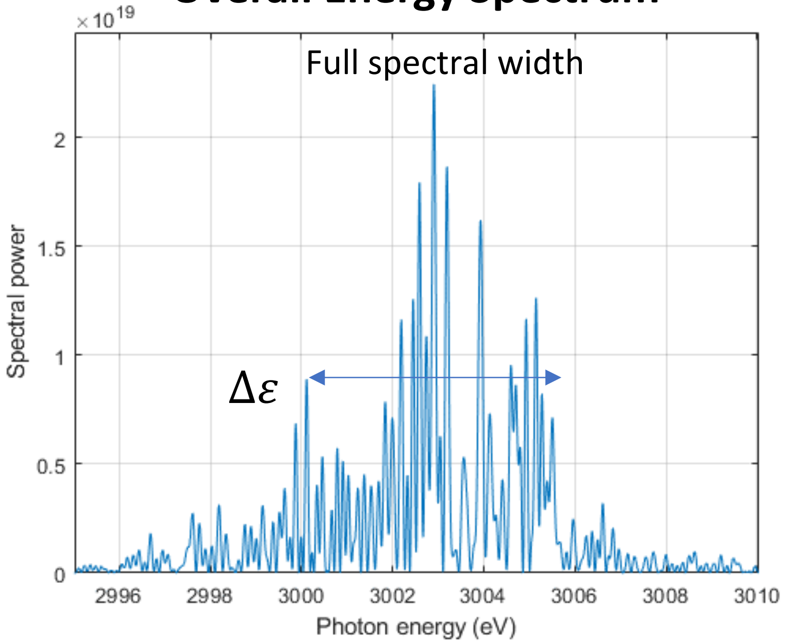
The radiation pulse width (electron bunch) is the Fourier conjugate of the individual spectral spike width. The longer the overall electron bunch (Δt) in the time domain, the narrower the spectral spike width ($\delta\varepsilon$) in the energy (frequency) domain.

SASE Time-Bandwidth (continued)

Expanded Time Scale



Overall Energy Spectrum



$$\Delta\varepsilon \cdot \delta t = 1.8\, eV \cdot fs$$

In the expanded time scale, the temporal spike width is the Fourier conjugate of the full spectral (energy) width. The shorter the temporal spikes (δt) in the time domain, the broader the overall spectral width ($\Delta\varepsilon$) in the energy domain.

SASE Bandwidth vs. z

The relative spectral bandwidth of a SASE FEL is plotted as a function of undulator length. The relative BW decreases along the undulator and reaches the minimum just before saturation.

Dependence of relative rms BW on z

$$\frac{\sigma_\omega(z)}{\omega} = 3\sqrt{2}\rho\sqrt{\frac{L_{g0}}{z}}$$

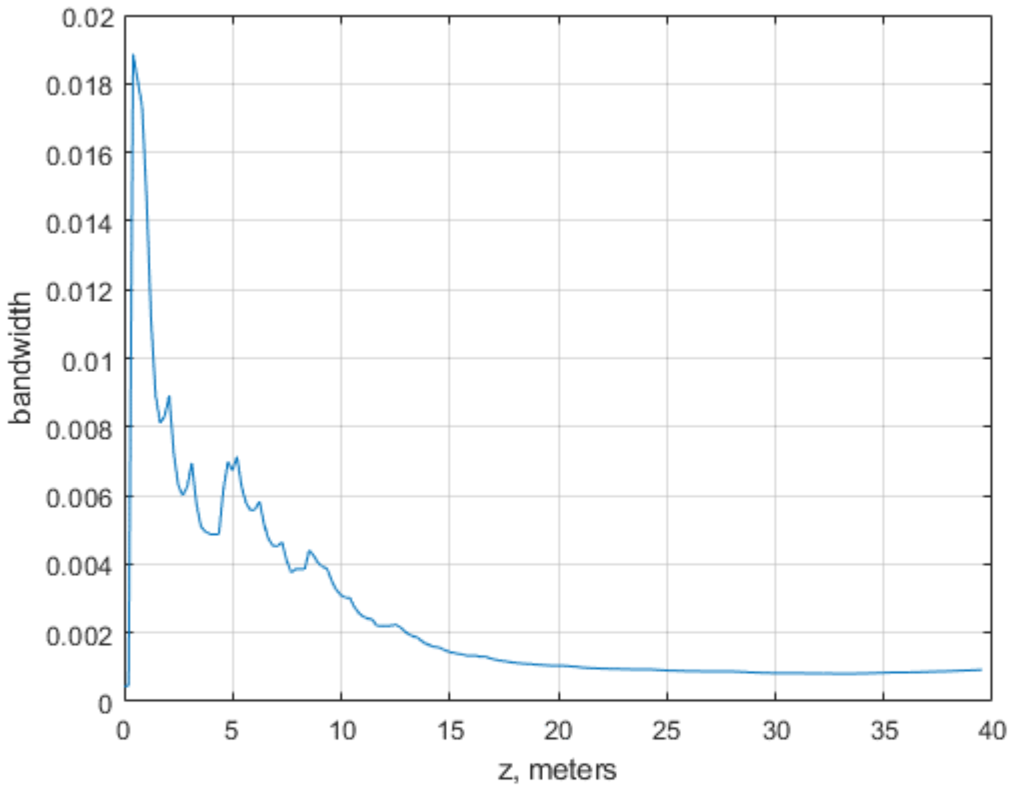
Relative BW (FWHM) $$\frac{\Delta\omega}{\omega} = 2\sqrt{ln2}\,\frac{\sigma_\omega(z)}{\omega}$$

Number of 1D gain lengths to reach saturation

$$L_{sat} = 21.8L_{g0}$$

Minimum BW (FWHM)

$$\frac{\Delta\omega}{\omega} \approx 1.5\rho$$



First Observations of SASE FEL

Microwave*	LLNL (1986)
Infrared	LANL (1997)
	UCLA-LANL-RRK-SLAC (1998) →
IR-Visible	UCLA-BNL-SLAC-LLNL (2001)
Visible-UV	ANL LEUTL (2001)
VUV	DESY (2000)
	BNL (2003)
Soft X-ray	DESY (2007)
Hard X-ray	SLAC (2009) →

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Measurements of Gain Larger than 10^5 at 12 μm in a Self-Amplified Spontaneous-Emission Free-Electron Laser

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First lasing and operation of an ångstrom-wavelength free-electron laser

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* The LLNL SASE experiment was done in a waveguide, not free space

SASE Amplitude Fluctuations

SASE pulse energy fluctuates significantly from pulse to pulse in the exponential growth regime, due to the stochastic nature of SASE shot noise.

$$\langle U_{pulse} \rangle = \text{Average pulse energy}$$

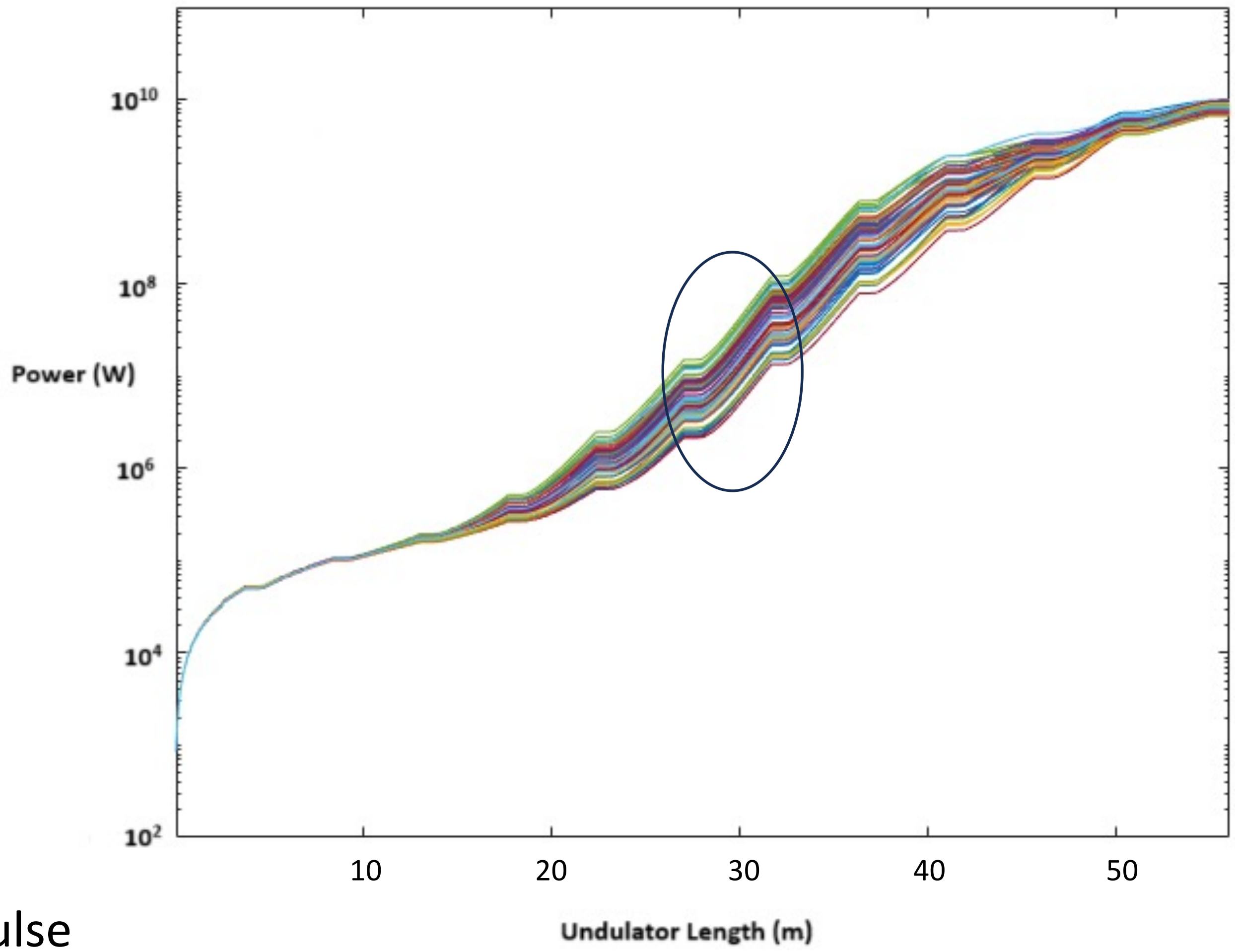
Define normalized pulse energy

$$u = \frac{U_{pulse}}{\langle U_{pulse} \rangle}$$

Fluctuation statistics of normalized amplitude

$$\sigma_u = \frac{1}{\sqrt{M}}$$

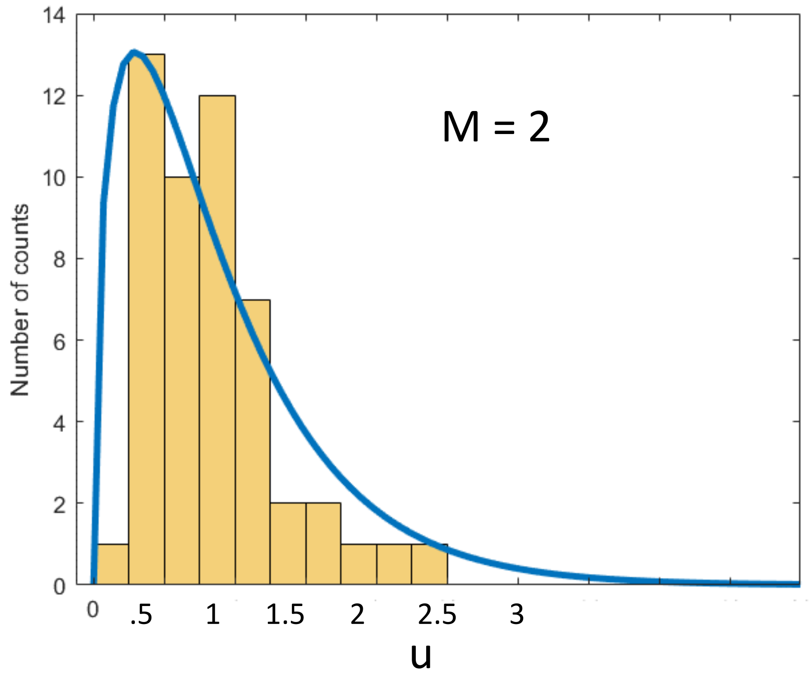
M = number of modes (spikes) in each SASE pulse

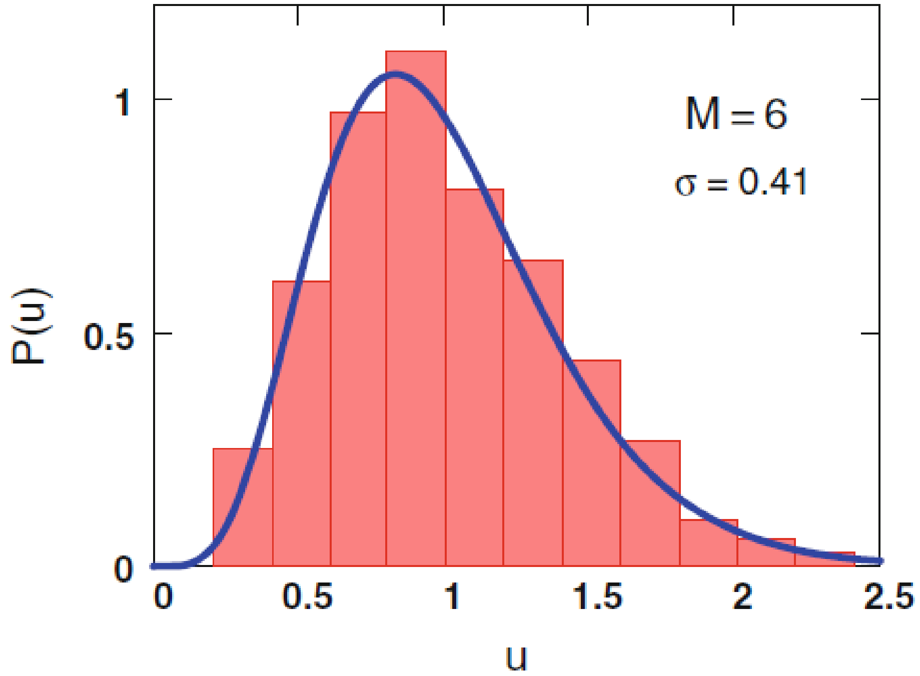


Poisson Statistics

Probability distribution of normalized pulse energy

$$p_M(u)du = \frac{M^M u^{M-1}}{\Gamma(M)} e^{-Mu} du$$





1D Theory of High-Gain FEL

Some Important Basic Concepts

- **FEL coupled first-order differential equations**

  A Normalized radiation field amplitude

  η Normalized electron energy modulation

  b Electron bunching

- **Slowly Varying Amplitude (SVA) approximation**

  Radiation field amplitude is a smooth and slowly varying function of z

$$\frac{\partial^2 \hat{E}}{\partial z^2} \ll k \frac{\partial \hat{E}}{\partial z}$$

- **Optical guiding**

  The radiation beam is guided by the high-current electron beams in the exponential gain regime

- **Third-order differential equation**

  For small η, combine the FEL coupled equations into a single third-order differential equation

- **Cubic dispersion equation** & **the three roots**

  For the resonant case and assuming solutions are of the form $\tilde{E}_x = Ae^{\alpha z}$, solve the cubic equation and obtain three roots

  Exponentially growing

  Exponentially decaying

  Oscillatory

FEL Coupled First-Order Equations

Normalized radiation field amplitude grows with electron microbunching

$$\frac{dA}{d\tau} = -\frac{1}{N}\sum_{n=1}^{N} \exp(-i\psi_n)$$

$$\frac{d\psi_n}{d\tau} = \bar{\eta}_n$$

$$\frac{d\eta_n}{d\tau} = -2\mathrm{Re}\left(Ae^{i\psi_n}\right)$$

Electron microbunching grows with the normalized electron energy modulation

Energy modulation grows with radiation field amplitude correlated with electron phase

Normalized FEL Variables

Normalized undulator coordinate

$$\tau = 2k_u \rho z$$

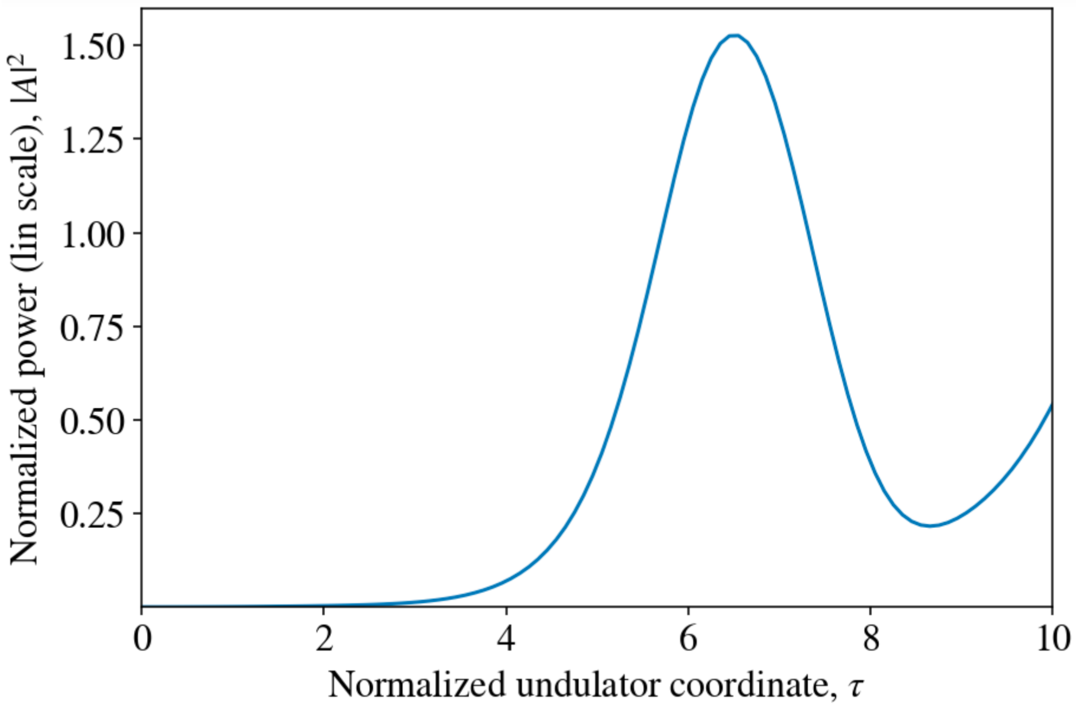
Normalized energy deviation from resonance

$$\bar{\eta}_n = \frac{\eta_n}{\rho}$$

Normalized radiation field amplitude

$$A = \frac{E}{E_s}$$

Saturation electric field $$E_s = \sqrt{\frac{Z_o \rho P_b}{\pi \sigma_r^2}}$$



Saturated normalized SASE power at zero initial energy detuning

$$|A|^2 \leq 1.5$$

Definitions of Important Variables

ψ_n : Phase of the n^{th} electron with respect to the FEL resonant radiation wave

η_n : Energy deviation of the n^{th} electron with respect to the FEL resonant dimensionless energy

$$\eta_n = \frac{\gamma_n - \gamma_r}{\gamma_r}$$

γ_r : Resonant dimensionless energy

$$\gamma_r = \sqrt{\frac{k_r}{2k_u}\left[1 + \frac{K^2}{2}\right]} \qquad k_r = \frac{2\pi}{\lambda_r}$$

Δ : Initial energy detuning from the FEL resonant energy

$\widehat{K}$: Undulator parameter corrected for the reduction due to figure-8 motion

$\tilde{J}_0$: Initial electron beam DC current density (A/m^2)

$\tilde{J}_1$: Transverse bunching current density at the fundamental wavelength

FEL Coupled First-Order Equations

Evolution of the n^{th} electron phase

$$\frac{d\psi_n}{dz} = 2k_u\eta_n$$

Radiation field amplitude grows with the first harmonic current density

$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c\widehat{K}}{4\gamma_R}\tilde{j}_1$$

First harmonic current density

$$\tilde{j}_1 = j_0\frac{2\pi}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Evolution of the n^{th} electron energy deviation

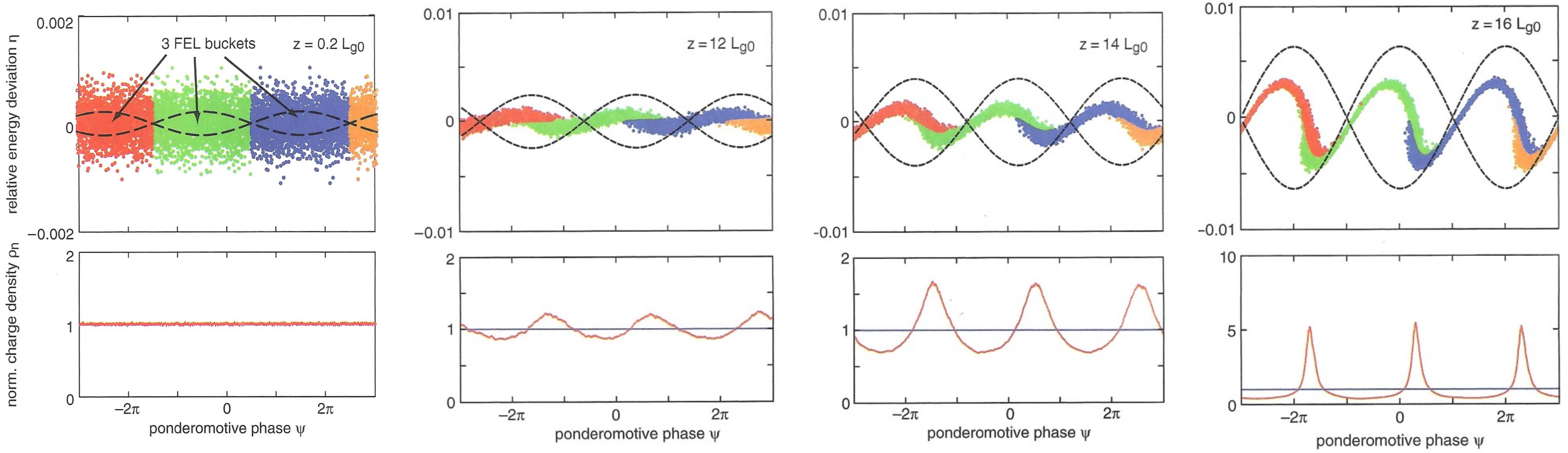
$$\frac{d\eta_n}{dz} = -\frac{e}{m_0c^2\gamma_R}\mathrm{Re}\left\{\left[\frac{\widehat{K}\tilde{E}_x}{2\gamma_R} - \tilde{E}_z\right]\exp(i\psi_n)\right\}$$

Radiation-electron interaction

Electron-electron interaction (space charge)

Space charge effects are negligible for FELs operating in the Compton regime (e.g., X-ray FELs). Space charge cannot be ignored for FELs operating in the Raman regime (e.g., THz FELs).

Evolution of Harmonic Current Density



Initial DC current density

$$j_0 = -n_e e c$$

$$n_e = \frac{N}{A_b \lambda_r}$$

$$j_0 = -ec\frac{k_r}{A_b}\frac{c_0}{2}$$

First harmonic current density is proportional to the electron bunching

$$j_1 = -ec\frac{k_r}{A_b}c_1$$

$$\tilde{j}_1 = j_0\frac{2\pi}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Phase Space Distribution Function, F

Consider a 2D phase-space density function with small 1st harmonic modulations.

$$F(\psi,\eta,z) = F_0(\eta) + Re\{\tilde{F}_1(\eta,z)\cdot e^{i\psi}\}$$

Assume F_0 is a Gaussian function of energy with small energy spread.

$$F_0(\eta) = \frac{1}{\sqrt{2\pi}\sigma_\eta} e^{-\frac{(\eta-\eta_0)^2}{2\sigma_\eta^2}}$$

The 1st harmonic current is related to 1st harmonic phase-space density

$$\tilde{j}_1 = j_0 \int_{-\delta}^{\delta} \tilde{F}_1(\eta,z)d\eta$$

Liouville's Theorem & Vlasov Equation

Liouville's equation governs the evolution of phase-space distribution with the independent coordinate. According to Liouville's Theorem, in the absence of dissipative force, the phase-space volume occupied by an ensemble of particles is conserved along the trajectory.

Generalized continuity equation (also known as Vlasov equation).

$$\frac{dF}{dz} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \psi}\frac{d\psi}{dz} + \frac{\partial F}{\partial \eta}\frac{d\eta}{dz} = 0$$

Rewrite the continuity equation for the 1st harmonic distribution function

$$\frac{\partial \tilde{F}_1}{\partial z} + i2k_u\tilde{F}_1 - \frac{e}{m_0c^2\gamma_R}\frac{dF_0}{d\eta}\left[\frac{\hat{K}\tilde{E}_x}{2\gamma_R} + \tilde{E}_z\right] = 0$$

Longitudinal Space Charge Field

Longitudinal space charge force is the repulsive force felt by an electron due to the presence of other electrons. This effect is important for FELs operating in the Raman regime such as THz FELs. It is negligible for FELs operating in the Compton regime, e.g., all X-ray FELs.

In the text book, the longitudinal space charge electric field is expressed as the first derivative of the transverse electric field with respect to z.

$$\tilde{E}_z = i\frac{4\gamma_r c}{\omega_r \widehat{K}}\frac{d\tilde{E}_x}{dz}$$

Rewrite the continuity equation for the 1st harmonic distribution function

$$\frac{\partial \tilde{F}_1}{\partial z} + i2k_u\tilde{F}_1 = \frac{e}{m_0c^2\gamma_R}\left[\frac{\widehat{K}\tilde{E}_x}{2\gamma_R} + i\frac{4\gamma_r c}{\omega_r \widehat{K}}\frac{d\tilde{E}_x}{dz}\right]\frac{dF_0}{d\eta}$$

Slowly Varying Amplitude Approximation

Treat the radiation as a 1D (no optical diffraction) wave equation driven by a complex transverse electron current along the x direction

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\tilde{E}(z,t) = \mu_0 \frac{\partial \tilde{J}_x}{\partial t}$$

Consider the following trial solution for an EM wave with complex amplitude that depends only on z

$$\tilde{E}(z,t) = \tilde{E}_x(z)e^{i(kz-\omega t)}$$

Insert the above trial solution into the wave equation and expand

$$\left(-k^2\tilde{E}_x(z) + 2ik\frac{d\tilde{E}_x(z)}{dz} + \frac{d^2\tilde{E}_x(z)}{dz^2} + \frac{\omega^2}{c^2}\tilde{E}_x(z)\right)e^{i(kz-\omega t)} = \mu_0\frac{\partial \tilde{J}_x}{\partial t}$$

Applying the SVA approximation, i.e., the second derivative is much smaller than the first derivative

$$\left(2ik\frac{d\tilde{E}_x(z)}{dz}\right)e^{i(kz-\omega t)} = \mu_0\frac{\partial \tilde{J}_x}{\partial t}$$

FEL Integro-differential Equation

$$\frac{d\tilde{E}_x(z)}{dz} = -\frac{\mu_0 c\hat{K}}{4\gamma_r}\tilde{j}_1 = j_0\int_{-\delta}^{\delta}\tilde{F}_1(\eta,z)d\eta$$

Integrate the continuity equation with respect to s, from s = 0 to $s = z$

$$\tilde{F}_1(\eta,z) = \frac{e}{m_e c^2\gamma_r}\int_0^z\left[\frac{\hat{K}\tilde{E}_x}{2\gamma_R} + i\frac{4\gamma_r c}{\omega_r\hat{K}}\frac{d\tilde{E}_x}{dz}\right]\frac{dF_0}{d\eta}\,e^{-i2k_u\eta\cdot(z-s)}ds$$

Integro-differential equation

$$\frac{d\tilde{E}_x(z)}{dz} = ik_u\frac{\mu_0\hat{K}n_e e^2}{m_e\gamma_r^2}\int_0^z\left[\frac{\hat{K}\tilde{E}_x}{2\gamma_R} + i\frac{4\gamma_r c}{\omega_r\hat{K}}\frac{d\tilde{E}_x}{dz}\right]h(z-s)ds$$

For a mono-energetic electron beam with initial energy detuning $\Delta = \gamma_0 - \gamma_r$

$$h(z-s) = (z-s)e^{-i2k_u\frac{\Delta}{\gamma_r}(z-s)}$$

Third-Order Equation

Detuning

Plasma wavenumber

$$\frac{\tilde{E}_x'''}{\Gamma^3} + 2i\frac{\eta}{\rho}\frac{\tilde{E}_x''}{\Gamma^2} + \left[\frac{k_p^2}{\Gamma^2} - \left(\frac{\eta}{\rho}\right)^2\right]\frac{\tilde{E}_x'}{\Gamma} - i\tilde{E}_x = 0$$

Gain parameter

$$\Gamma = \left[\frac{\pi \hat{K}^2 I_p}{4\gamma_r^3 \lambda_u \sigma^2 I_A}\right]^{\frac{1}{3}}$$

$$\rho = \frac{\Gamma}{2k_u}$$

Prime denotes full derivatives with respect to z

$$\tilde{E}_x'(z) = \frac{d\tilde{E}_x(z)}{dz}$$

$$\tilde{E}_x''(z) = \frac{d^2\tilde{E}_x(z)}{dz^2}$$

$$\tilde{E}_x'''(z) = \frac{d^3\tilde{E}_x(z)}{dz^3}$$

Special case:

1. Beam energy does not deviate significantly from the resonant energy
2. X-ray FEL (Compton regime)

The second and third terms vanish for this case, and the third order equation reduces to

$$\frac{\tilde{E}_x'''}{\Gamma^3} - i\tilde{E}_x = 0$$

Cubic Equation & the Three Roots

Applying the resonant condition to an FEL operating in the Compton regime, and assuming solution of the form $\tilde{E}_x(z) = A\mathrm{e}^{\alpha z}$, we obtain the cubic equation

$$\alpha^3 = i\Gamma^3$$

Gain parameter

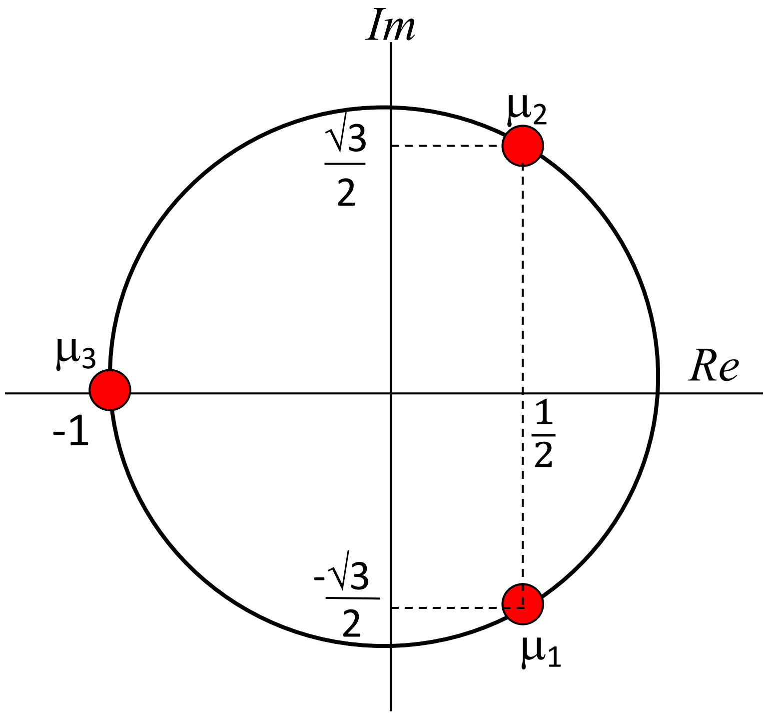
$\Gamma = 2k_u\rho$

The three roots of the cubic equation:

$$\alpha_1 = i\mu_1\Gamma = \frac{(i+\sqrt{3})}{2}\Gamma$$ Exponentially growing mode

$$\alpha_2 = i\mu_2\Gamma = \frac{(i-\sqrt{3})}{2}\Gamma$$ Exponentially decaying mode

$$\alpha_3 = i\mu_3\Gamma = -i\Gamma$$ Oscillatory mode



General Solutions & the A Matrix

Write the solution as a linear combination of the eigen-functions $V_j = e^{\alpha_j z}$

$$\tilde{E}_x(z) = c_1 V_1 + c_2 V_2 + c_3 V_3$$

where c_1, c_2 and c_3 are the coefficients of the linear combination. Taking the derivatives of the eigen-functions and expressing them in terms of the $\boldsymbol{A}$ matrix, we arrive at the initial conditions given below

$$\begin{array}{r} \text{initial radiation electric field} \\ \\ \text{initial bunching} \\ \\ \text{initial energy modulation} \end{array} \begin{pmatrix} \tilde{E}_x(0) \\ \\ \tilde{E}'_x(0) \\ \\ \tilde{E}''_x(0) \end{pmatrix} = \boldsymbol{A} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix}$$

Finding the General Solution Coefficients

The general solution can be expressed as a linear combination of eigenfunctions

$$\tilde{E}_x(z) = c_1 V_1 + c_2 V_2 + c_3 V_3$$

where the eigenfunctions are $$V_j = e^{\alpha_j z}$$

The coefficients of the general solution can be calculated by applying the inverse A matrix to the initial condition vector.

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boldsymbol{A}^{-1} \cdot \begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix}$$

Use A^{-1} matrix to calculate c_1, c_2, and c_3 coefficients from the initial conditions

Resonant Case and Zero Energy Spread

Eigenvalues (roots of cubic equation)

$$\alpha_1 = \frac{\left(i+\sqrt{3}\right)}{2}\Gamma$$

$$\alpha_2 = \frac{\left(i-\sqrt{3}\right)}{2}\Gamma$$

$$\alpha_3 = -i\Gamma$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ \left(i+\sqrt{3}\right)\Gamma/2 & \left(i-\sqrt{3}\right)\Gamma/2 & -i\Gamma \\ \left(i+\sqrt{3}\right)^2\Gamma^2/4 & \left(i-\sqrt{3}\right)^2\Gamma^2/4 & -\Gamma^2 \end{pmatrix}$$

Invert the above $\boldsymbol{A}$ matrix

$$\boldsymbol{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & \left(\sqrt{3}-i\right)/(2\Gamma) & \left(-i\sqrt{3}+1\right)/(2\Gamma^2) \\ 1 & \left(-\sqrt{3}-i\right)/(2\Gamma) & \left(i\sqrt{3}+1\right)/(2\Gamma^2) \\ 1 & i/\Gamma & -1/\Gamma^2 \end{pmatrix}$$

Seeding with an External Laser

The initial condition vector of a externally seeded FEL involves a coherent radiation electric field and an initially unbunched (with no density modulation) electron beam at the undulator entrance

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}'_x(0) \\ \tilde{E}''_x(0) \end{pmatrix} = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix}$$

Radiation electric field

Zero initial bunching

Zero initial energy modulation

Coefficients of the solution

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boldsymbol{A}^{-1} \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} E_0 \\ E_0 \\ E_0 \end{pmatrix}$$

High-gain Seeded FEL

Complex electric field versus z

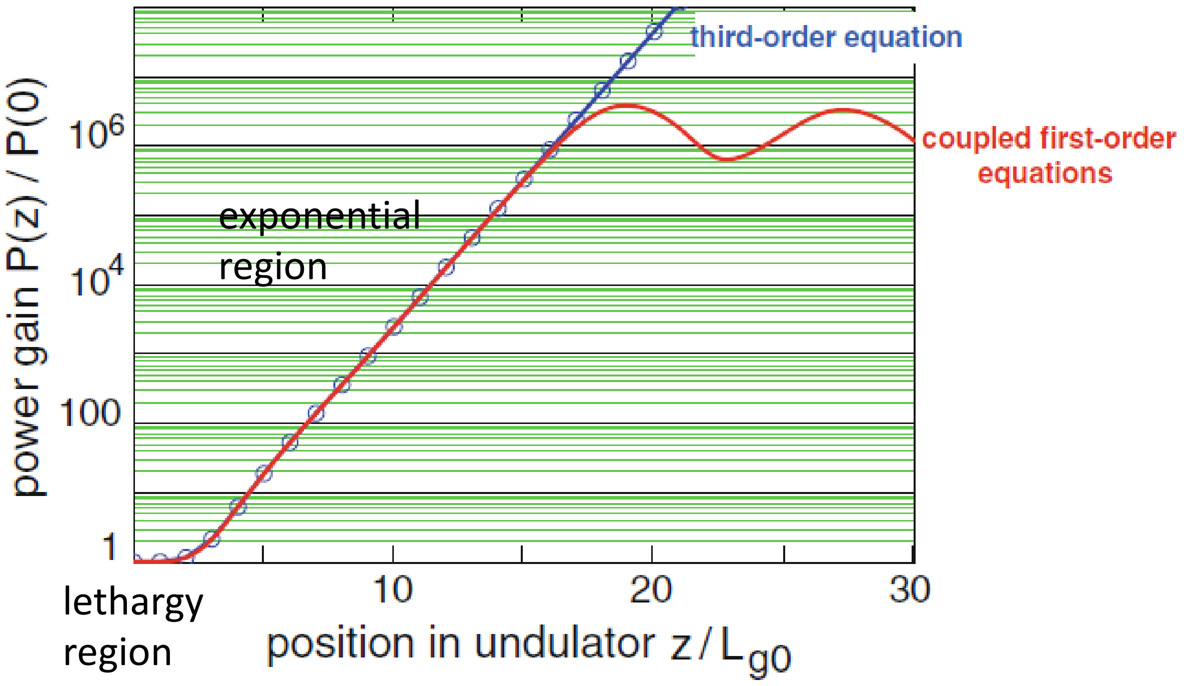
$$\tilde{E}_x(z) = \frac{E_0}{3} \cdot \left[\exp\left(\frac{(i+\sqrt{3})\Gamma}{2}\mathrm{z}\right) + \exp\left(\frac{(i-\sqrt{3})\Gamma}{2}\mathrm{z}\right) + \exp(-i\Gamma\mathrm{z})\right]$$

FEL intensity versus z in the exponential regime

$$\left|\tilde{E}_x(z)\right|^2 = \frac{E_0^2}{9} e^{\sqrt{3}\,\Gamma z}$$

Gain parameter $\qquad \Gamma = 2k_u\rho$

In the lethargy region (~2 L_{g0}) the three roots interfere with one another and the radiation power does not grow or grows slowly with z



SASE FEL Starting with Noise

SASE FEL initial condition involves a beam of monoenergetic electrons with start-up shot noise due to the electron's discrete nature as the seed.

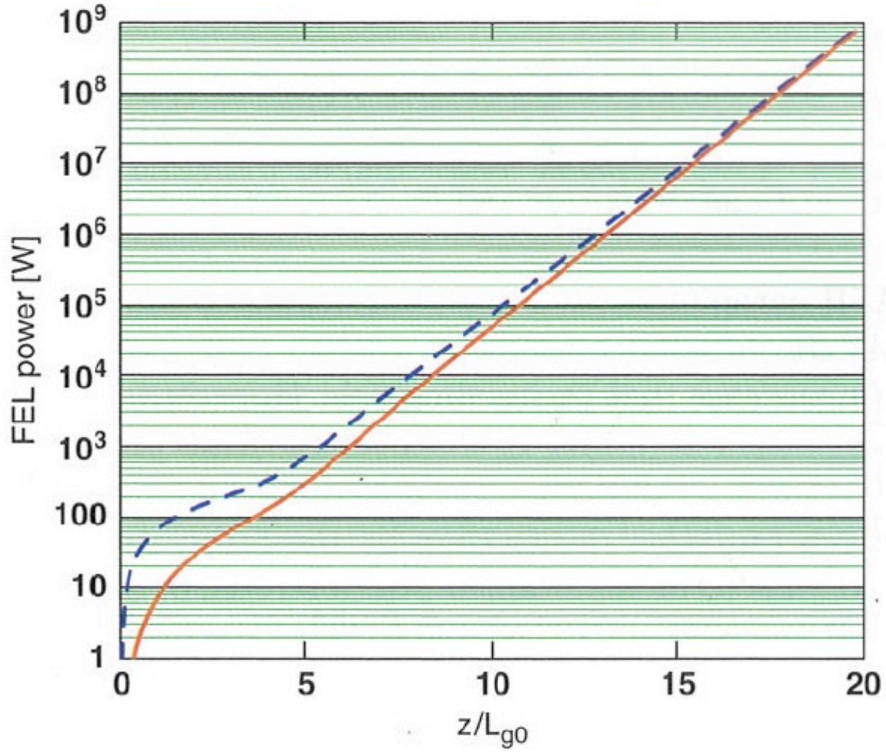
$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}'_x(0) \\ \tilde{E}''_x(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \frac{\mu_0 c \hat{K}}{4\gamma_r} \tilde{j}_1(0)$$

Equivalent current density

$$\tilde{j}_1(0) = \frac{1}{A_b}\sqrt{\frac{eI_0}{\pi}\Delta\omega}$$

Initial bandwidth is ~ twice beam energy spread

$$\left(\frac{\sigma_\omega}{\omega}\right)_{noise} = 2\left(\frac{\sigma_\gamma}{\gamma}\right)_{e-beam}$$



The solid (red) line corresponds to power for a constant bandwidth and the dash (blue) line corresponds to power for a variable bandwidth.

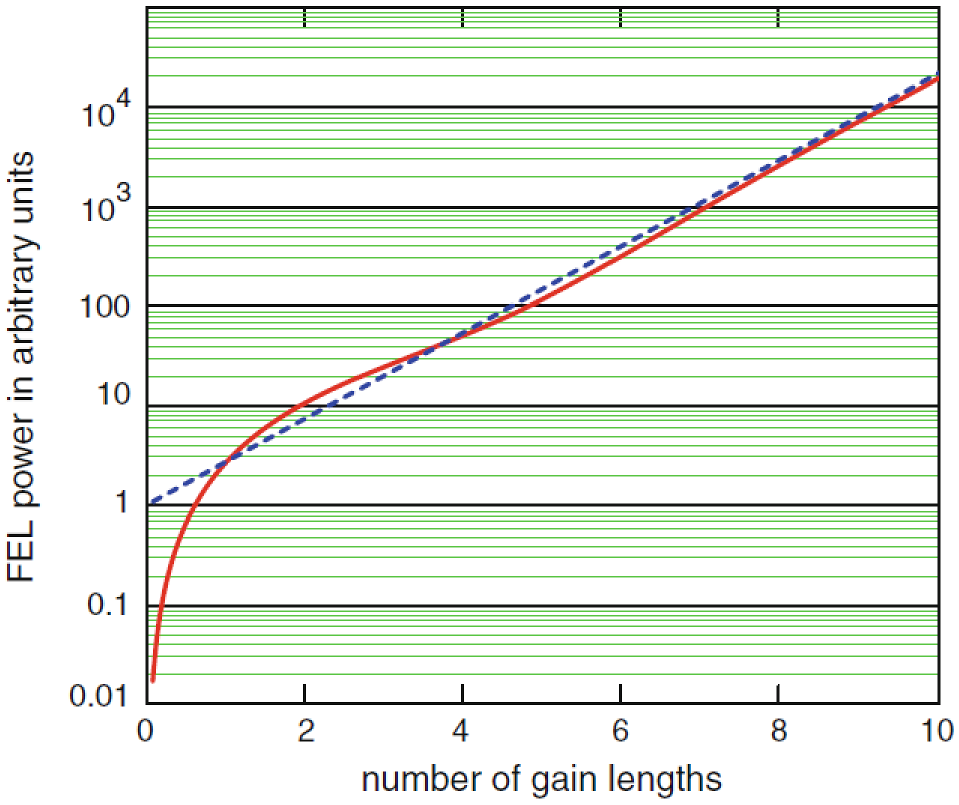
FEL Seeded with Bunched Beams

Another initial condition is when the electrons are periodically bunched before injected into the undulator, with zero initial radiation power.

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ i2k_u\eta \end{pmatrix} \frac{\mu_0 c\hat{K}}{4\gamma_r}\tilde{j}_1(0)$$

Initially, the electron beam has density modulations with a period equal to the radiation wavelength. The radiation power starts out zero, but rises to the equivalent seed power as given below

$$P_{eq}(0) \approx \rho P_b b^2(0)$$



Three-dimensional Effects

Beam Optics (Twiss) Functions

Beta function* (beam size)

$$\beta_x = \frac{\langle x^2 \rangle}{\varepsilon_x}$$

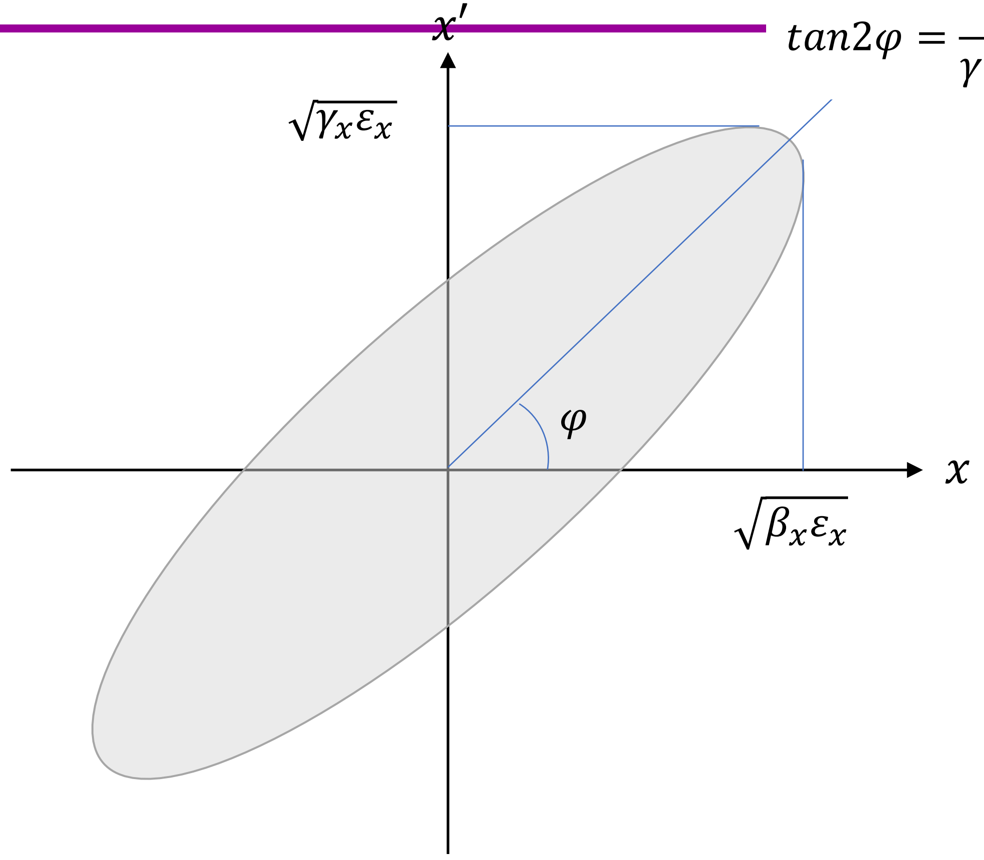
Gamma function (beam divergence)

$$\gamma_x = \frac{\langle x'^2 \rangle}{\varepsilon_x}$$

Alpha function (phase-space angle)

$$\alpha_x = \frac{-\langle x x' \rangle}{\varepsilon_x} = -\frac{1}{2}\frac{d\beta_x}{dz}$$

$$\tan 2\varphi = \frac{2\alpha}{\gamma - \beta}$$



$$\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2 = \varepsilon_x$$

* Note β_x is the Twiss beta function, not x velocity relative to the speed of light.

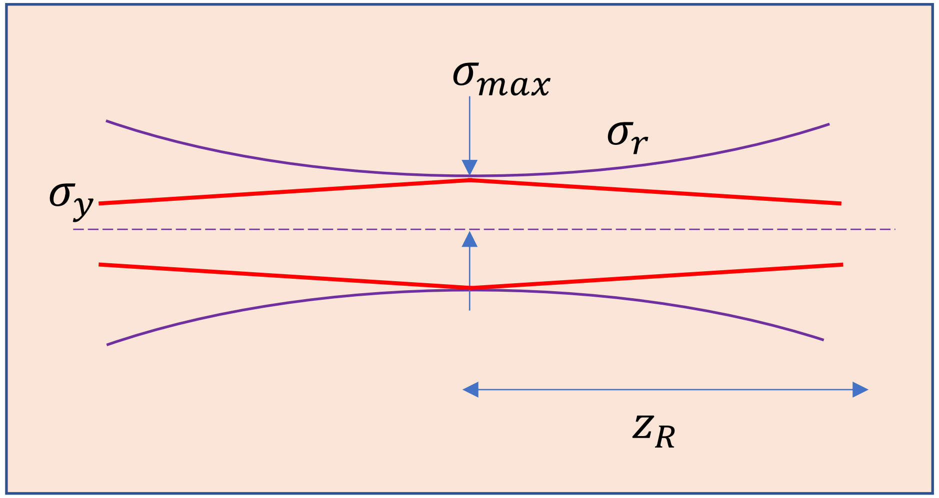
Focusing β and Rayleigh Range

For the electron beam to efficiently transfer its energy to the radiation beam, the electron beam un-normalized emittance must be smaller than the photon beam emittance, i.e.,

$$\frac{4\pi\varepsilon_u}{\lambda_r} \leq 1$$

This stringent condition is not satisfied in most hard X-ray FELs. The 3D effect due to emittance shows up as large angles in the electron beam as it traverses the FODO lattice. To minimize this effect, the FODO lattice is designed with focusing β larger than the 1D gain length, i.e., $\frac{L_{1D}}{\beta_{ave}} < 1$

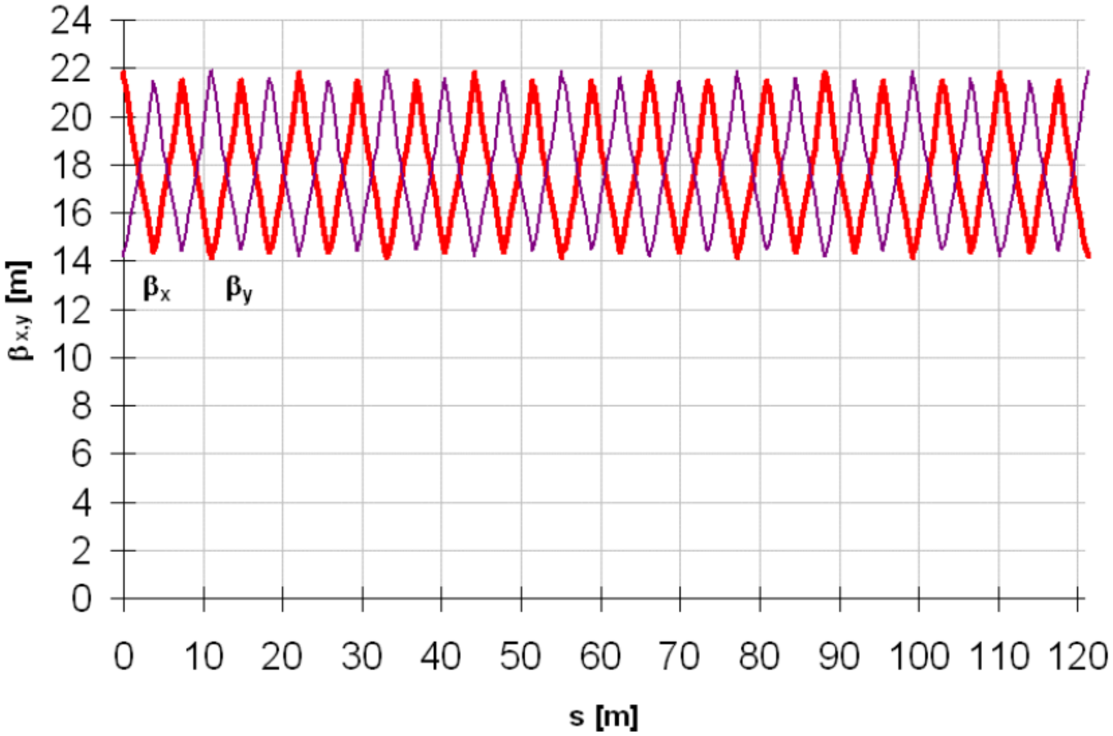
$$\sigma_{max} = \sqrt{\varepsilon_{x,y}\beta_{max}\left(1 + \frac{z^2}{\beta_{max}^2}\right)}$$

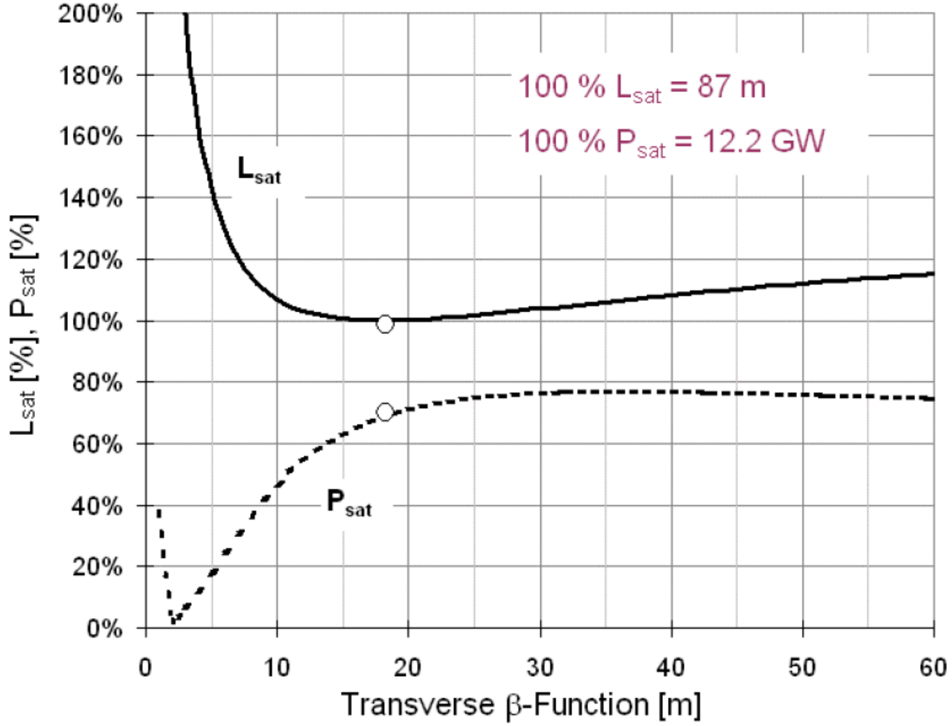


$$\sigma_r = \sqrt{\frac{\lambda}{4\pi} z_R\left(1 + \frac{z^2}{z_R^2}\right)}$$

$$\beta_{ave} \sim z_R$$

Optimum Focusing β Function





$$\beta_{ave} \sim \frac{1}{2}(\beta_{max} + \beta_{min})$$

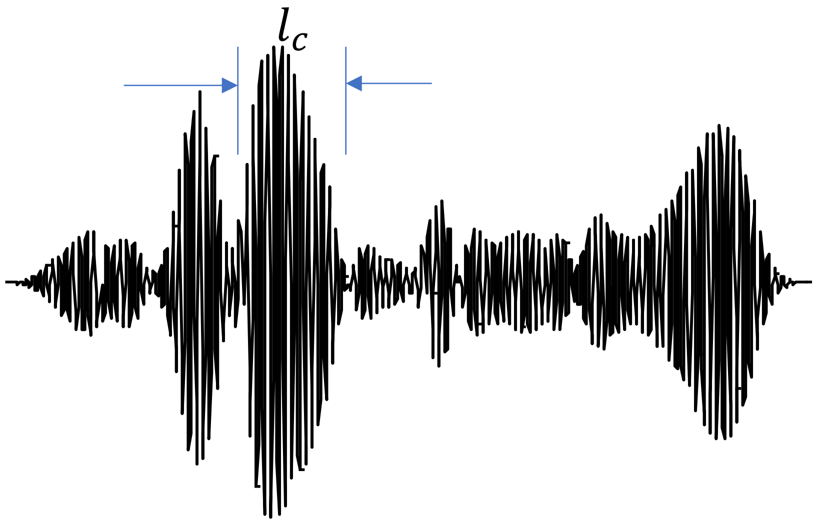
Too short β functions increase the angular modulations, thus increase the electron beam's effective energy spread, resulting in lower power. Too long β functions reduce the current density and also lead to lower power. Note the reduction is gradual beyond the optimum β function.

Electron Beam Energy Spread

Electrons must maintain the same axial velocity during the coherence length l_c

$$l_c \approx N_G \lambda$$

$$N_G \approx \frac{1}{4\pi\rho}$$



SASE coherence length is approximately the slippage length over the 1D gain length

The initial relative beam energy spread must be less than ρ

$$\frac{\sigma_\gamma}{\gamma} \leq \rho \qquad \frac{\sigma_\gamma}{\gamma} \leq \frac{1}{4\pi N_G}$$

Ming-Xie Parameterization – Part 1

	Ming-Xie parameters	Conditions for 1D Theory
Diffraction		
$L_{1D} \leq z_R$	$\eta_d = \frac{L_{1D}}{z_R}$	$\eta_d < 1$ $\boxed{z_R > L_{1D}}$
Emittance		
$\frac{L_{1D}}{\beta_{ave}} \frac{4\pi\varepsilon_u}{\lambda_r} \leq 1$	$\eta_\varepsilon = \frac{L_{1D}}{\beta_{ave}} \frac{4\pi\varepsilon_u}{\lambda_r}$	$\eta_\varepsilon < 1$ $\boxed{\beta_{ave} > L_{1D}}$
Energy spread		
$\frac{\sigma_\gamma}{\gamma} \leq \frac{1}{4\pi N_G}$	$\eta_\gamma = \frac{4\pi L_{1D}}{\lambda_u} \frac{\sigma_\gamma}{\gamma}$	$\eta_\gamma < 1$ $\boxed{\frac{\sigma_\gamma}{\gamma} < \rho}$

Ming-Xie Parameterization – Part 2

3D effects increase the power gain length by a factor $F(\eta_d, \eta_\varepsilon, \eta_\gamma) = 1 + \Lambda(\eta_d, \eta_\varepsilon, \eta_\gamma)$

$$\Lambda(\eta_d, \eta_\varepsilon, \eta_\gamma) = a_1 {\eta_d}^{a_2} + a_3 {\eta_\varepsilon}^{a_4} + a_5 {\eta_\gamma}^{a_6} \\ + a_7 {\eta_\varepsilon}^{a_8} {\eta_\gamma}^{a_9} + a_{10} {\eta_d}^{a_{11}} {\eta_\gamma}^{a_{12}} + a_{13} {\eta_d}^{a_{14}} {\eta_\varepsilon}^{a_{15}} + a_{16} {\eta_d}^{a_{17}} {\eta_\varepsilon}^{a_{18}} {\eta_\gamma}^{a_{19}}$$

a_1=0.45	a_2=0.57	a_3=0.55	a_4=1.6
a_5=3	a_6=2	a_7=0.35	a_8=2.9
a_9=2.4	a_{10}=51	a_{11}=0.95	a_{12}=3
a_{13}=5.4	a_{14}=0.7	a_{15}=1.9	a_{16}=1140
a_{17}=2.2	a_{18}=2.9	a_{19}=3.2	

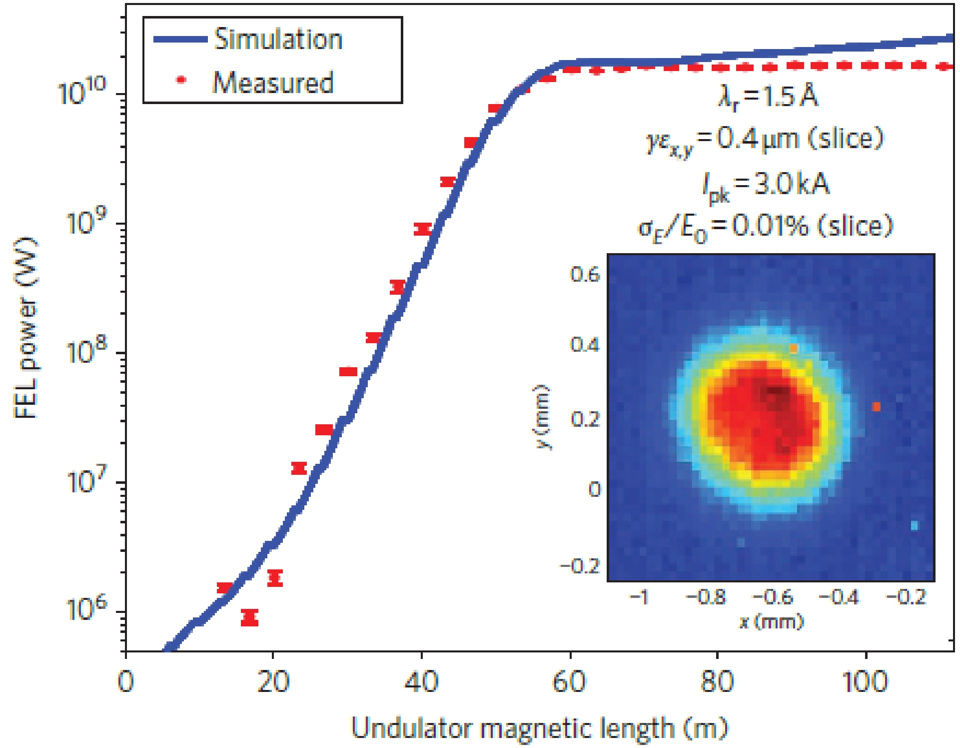
3D Power gain length $L_{G,3D} = L_{g0}(1 + \Lambda)$

3D Saturated power $P_{sat,3D} = \dfrac{\rho P_b}{(1 + \Lambda)^2}$

Comparing MX, Genesis to Data

Parameters	Symbol	Value
Beam energy	E_b	13.6 GeV
Peak current	I_p	3.0 kA
Slice emittance	$\varepsilon_{n,rms}$	0.4 µm
rms energy spread	$\sigma_{\gamma,rms}/\gamma$	0.01%
FEL wavelength	λ	1.5 Å
FEL parameter	ρ	7 x 10^{-4}
1D gain length	L_{g0}	2.0 m
1D Saturated power	P_{sat}	28 GW

	Ming-Xie	Genesis	LCLS Data
3D effect, Λ	0.21		
Power gain length	2.4 m	3.6 m	3.5 m
Saturated power	20 GW	20 GW	15 GW



LCLS experimental data from “First lasing and operation of an angstrom-wavelength free-electron laser” P. Emma et al., Nature Photonics **4**, 641–647(2010)

Summary of SASE & High-Gain FEL

- SASE starts from noise, grows exponentially along a very undulator and saturates at peak power of 10s of GW. The SASE x-ray pulses only have partial temporal coherence and consist of multiple sub-femtosecond spikes, each with its own coherence length. The SASE x-ray FEL output has significant pulse-to-pulse energy and spectral fluctuations.

- The 1D FEL theory is based on interaction between a mono-energetic electron beam and a paraxial radiation beam under SVA approximation. This interaction is described by three coupled equations involving the radiation field amplitude, electron bunching and energy detuning.

- For small energy detuning, we combine the three first-order equations into a single third-order equation that gives rise to the cubic equation with three roots. One of these roots corresponds to the mode that grows exponentially along z with a characteristic 1D gain length.

- The effects of diffraction, emittance and energy spread can be analyzed using the Ming-Xie parametrization approach that provides estimates of the 3D gain length and saturated power.

FEL Bucket & Synchrotron Oscillations

Radiation electric field amplitude

$$E_0 = \sqrt{2Z_0 I_r} = \sqrt{\frac{2I_0}{c\epsilon_0}}$$

FEL bucket half-height

$$\eta_{max} = \sqrt{\frac{eE_0K}{k_u m_e c^2}}$$

Synchrotron oscillation period

$$L_S = \frac{\lambda_u}{2\eta_{max}}$$