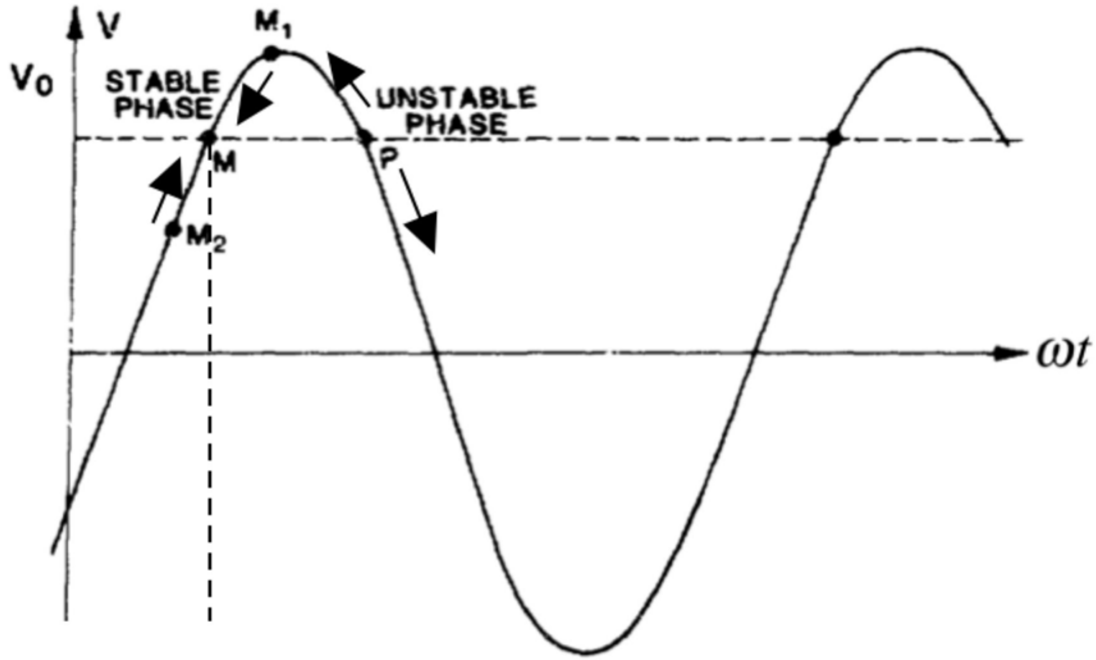
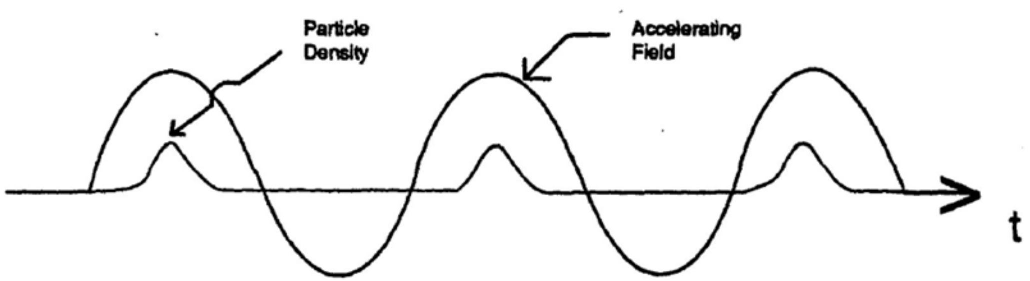
Phase Stability Principle : Stable and Unstable Phases

RF phase of synchronous particle is selected to be when the field is increasing in time. Earlier particle receive smaller energy kick than the synchronous one and will be slowing down with respect to synchronous particle. Particles, which arrive later to accelerating gap, receive larger energy gain, and will run down the synchronous particle. When non-equilibrium particles exchange their positions, this process is repeated for new particles setup, which results in stable longitudinal oscillations around synchronous particle. While synchronous particle monotonically increases it's energy, other particle perform oscillation around synchronous particle, and also increase their energy. Such principle is called resonance principle of particle acceleration.

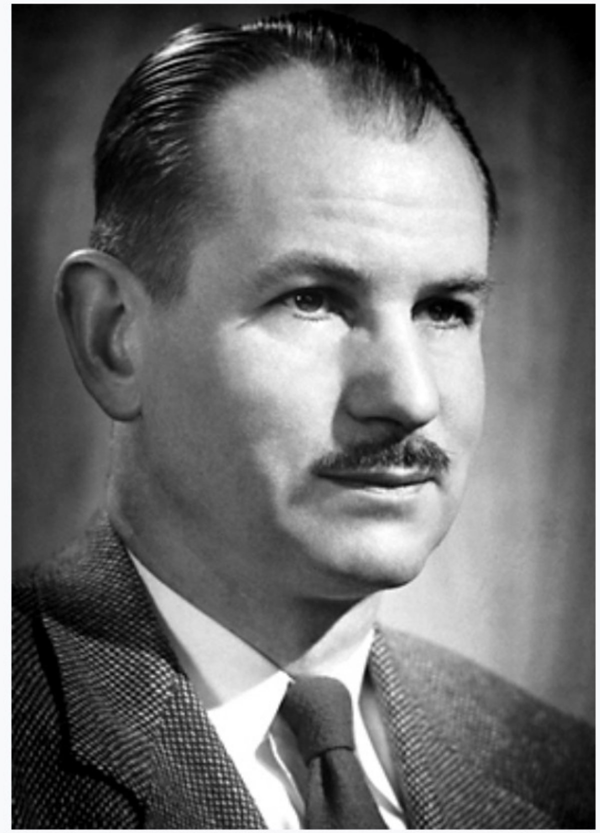




Discovery of Phase Stability Principle (Autophasing)

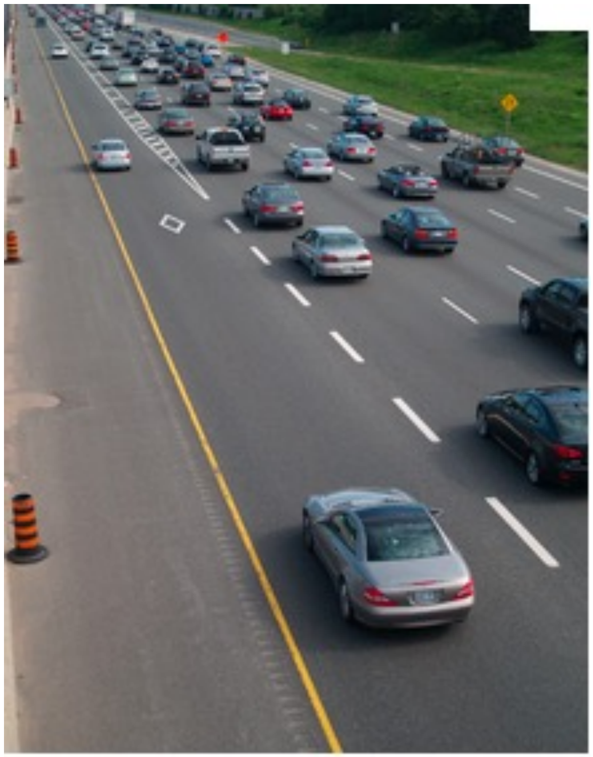
Vladimir Iosifovich Veksler

Edwin McMillan



Vladimir Veksler (1944) at the Lebedev Institute of Physics and later Edwin McMillan (1945) at the University of California, Berkeley, independently discover the principle of phase stability, a cornerstone of modern accelerators.

Beam Bunching: Analogy with Traffic

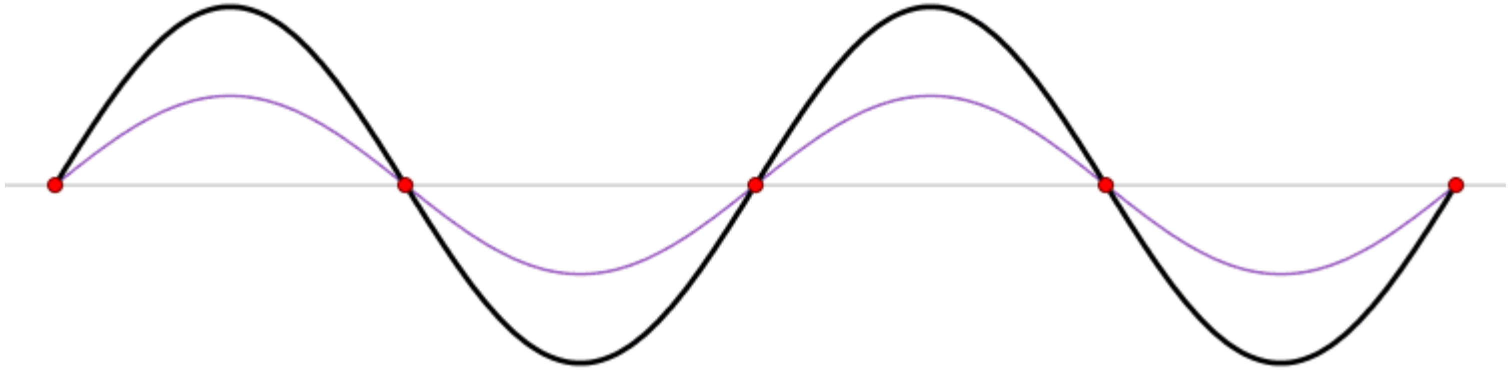


Continuous traffic

Bunched car traffic created by a traffic light

Standing Wave as a Combination of Traveling Waves

$$E\cos(k_z z)\cos(\omega t) = \frac{E}{2}[\cos(\omega t - k_z z) \quad + \quad \cos(\omega t + k_z z)]$$



Accelerating Wave

Increment of energy of arbitrary particle in RF gap

$$\Delta W = q \int_{-L/2}^{L/2} E_g(z,r)\,\cos(\omega t)\,dz$$

The RF phase at the time of arrival of arbitrary particle in point with coordinate z

$$\omega t(z) = \varphi + k_z z$$

Standing wave can be represented as combination of traveling waves:

$$\sum_{m=1}^{\infty}\cos(\frac{2\pi m z}{L})\cos(\omega t) = \frac{1}{2}\sum_{m=1}^{\infty}\cos(\omega t - \frac{2\pi m z}{L}) + \frac{1}{2}\sum_{m=1}^{\infty}\cos(\omega t + \frac{2\pi m z}{L})$$

traveling waves in z – direction

traveling waves in opposite direction

Only m = 1 harmonic of traveling waves propagating in z-direction contributes to energy gain of particle. In general case $m = n$ *(where $L = n\beta\lambda$).*

$$\int_{-L/2}^{L/2}\cos(\frac{2\pi z}{L} - \frac{2\pi m z}{L} + \varphi)dz = \begin{cases} L\cos\varphi, & m = 1 \\ 0, & m \neq 1 \end{cases}$$

$$\int_{-L/2}^{L/2}\cos(\frac{2\pi z}{L} + \frac{2\pi m z}{L} + \varphi)dz = 0$$

Accelerating Wave (cont.)

Increment of energy of arbitrary particle in RF gap

$$\Delta W = q E_o L T I_o(\frac{2\pi r}{\beta\gamma\lambda}) \cos\varphi$$

Taking into account equation for increment of particle energy $dW/dz = qE_z(z,r,t)$, the equivalent accelerating traveling wave is

$$E_z = E_o T I_o(\frac{k_z r}{\gamma}) \cos\varphi$$

Amplitude of equivalent traveling wave

$$E = E_o T$$

Electromagnetic field of accelerating wave

$$E_z = E I_o(\frac{k_z r}{\gamma}) \cos\varphi$$

$$E_r = -\gamma E I_1(\frac{k_z r}{\gamma}) \sin\varphi$$

$$B_\theta = -\frac{\beta\gamma}{c} E I_1(\frac{k_z r}{\gamma}) \sin\varphi$$

Longitudinal Dynamics in Accelerating Wave

Accelerating wave propagates with velocity

$$\beta_{ph} c = \frac{dz}{dt}$$

Synchronous particle is the one, which velocity instantaneously coincides with that of the accelerating wave

$$\boxed{\beta_s = \beta_{ph}}$$

Integration of equation

$$t = \frac{\varphi}{\omega} + \int_0^z \frac{dz}{\beta_{ph} c}$$

gives the phase of particle with respect to accelerating wave

$$\varphi = \omega t - \int^z k_z \, dz$$

where wave number is

$$k_z(z) = \frac{2\pi}{\beta_{ph}(z)\lambda} = \frac{2\pi}{\beta_s \lambda}$$

Phase velocity of accelerating wave (velocity of synchronous particle) is determined by condition φ = const:

$$d\varphi = \omega dt - k_z dz = 0$$

$$\boxed{\beta_{ph} = \beta_s = \frac{\omega}{c\, k_z}}$$

For arbitrary particle:

$$\frac{d\varphi}{dz} = \omega \frac{dt}{dz} - k_z$$

Longitudinal equations of motion of arbitrary particle

$$\frac{d\varphi}{dz} = \frac{2\pi}{\lambda}\left(\frac{1}{\beta} - \frac{1}{\beta_{ph}(z)}\right)$$

$$\frac{dW}{dz} = qE\cos\varphi$$

Longitudinal Dynamics in Accelerating Wave

Dependence $\beta_{ph}(z)$ is determined by geometry of accelerating structure. For synchronous particle $\beta = \beta_{ph}(z)$. Synchronous phase has negative value with respect to peak of the field.

$$\frac{d\varphi}{dz} = \frac{2\pi}{\lambda}\left(\frac{1}{\beta} - \frac{1}{\beta_{ph}(z)}\right)$$

$$\frac{dW}{dz} = qE\cos\varphi$$

Phase of particle with $\beta > \beta_{ph}$ becomes more negative, and such particle is slowing down with respect to synchronous particle. Correspondingly, particle with $\beta < \beta_{ph}$ is accelerating with respect to synchronous particle. Therefore, particles perform oscillations around synchronous particle. Synchronous phase is established inevitably in a channel with certain dependence $\beta_{ph}(z)$ and certain value of accelerating field E (*autophasing principle*):

$$\cos\varphi_s = \frac{1}{qE}\frac{dW_s}{dz}$$

where change of energy of synchronous particle $\frac{dW_s}{dz} = mc^2\beta_{ph}\gamma_{ph}^3\frac{d\beta_{ph}}{dz}$

With variation of field E, synchronous phase is changing, and particles start oscillate around new synchronous phase. Therefore, synchronous phase is entirely determined by the accelerating channel.

Oscillations Around Synchronous Particle

Equations of longitudinal motion in traveling wave near axis $I_o(\frac{k_z r}{\gamma}) \approx 1$

$$\frac{dp_z}{dt} = qE\cos\varphi$$

$$\frac{dz}{dt} = \frac{p_z}{m\gamma}$$

Longitudinal momentum deviation from synchronous particle

$$p_\zeta = p_z - p_s$$

Deviation from synchronous particle

$$\zeta = z - z_s$$

Phase of particle in traveling wave:

$$\varphi = \omega t - k_z(z_s + \zeta) = \varphi_s - k_z\zeta$$

Equations of particle motion around synchronous particle

$$\frac{d\zeta}{dt} = \frac{dz}{dt} - \frac{dz_s}{dt} = \Delta(\beta c) \to d(\beta c)$$

$$d\beta = \frac{1}{\gamma^3}\frac{dp}{mc}$$

$$\boxed{\frac{dp_\zeta}{dt} = qE[\cos(\varphi_s - k_z\zeta) - \cos\varphi_s]}$$

$$\boxed{\frac{d\zeta}{dt} = \frac{p_\zeta}{m\gamma^3}}$$

Hamiltonian of Longitudinal Oscillations

Equations of motion around synchronous particle can be derived from Hamiltonian

$$H = \frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s]$$

Hamiltonian equations of motion:

$$\frac{d\zeta}{dt} = \frac{\partial H}{\partial p_\zeta} \qquad \frac{dp_\zeta}{dt} = -\frac{\partial H}{\partial \zeta}$$

Hamiltonian describes particle oscillations around synchronous particle, where parameters γ, *E*, k_z depend on longitudinal position. Let us assume that parameters γ, *E*, k_z, are changing slowly during particle oscillations. Hamiltonian with constant values of γ, *E*, k_z, is a *constant of motion*. Actually, in this case:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \zeta}\frac{d\zeta}{dt} + \frac{\partial H}{\partial p_\zeta}\frac{dp_\zeta}{dt} = \frac{\partial H}{\partial \zeta}\frac{\partial H}{\partial p_\zeta} - \frac{\partial H}{\partial p_\zeta}\frac{\partial H}{\partial \zeta} = 0$$

Time-independent Hamiltonian *coincides with particle energy (kinetic + potential)*. Equation $dH/dt = 0$ expresees conservation of energy in isolated system (conservative approximation). In this case, we get equation for phase space trajectory $p_\zeta = p_\zeta(\zeta)$ as equation

$$H(\zeta, p_\zeta) = const$$

Hamiltonian of Longitudinal Oscillations in ($\Delta W, \psi$)

Another pair of canonical variables: $\psi = \varphi - \varphi_s$, $\Delta W = W_s - W$

Phase deviation from synchronous particle $$\psi = -k_z \zeta$$

Inverse energy deviation from synchronous particle: $$\Delta W = -\beta c p_\zeta$$

Hamiltonian of energy-phase oscillations around synchronous particle:

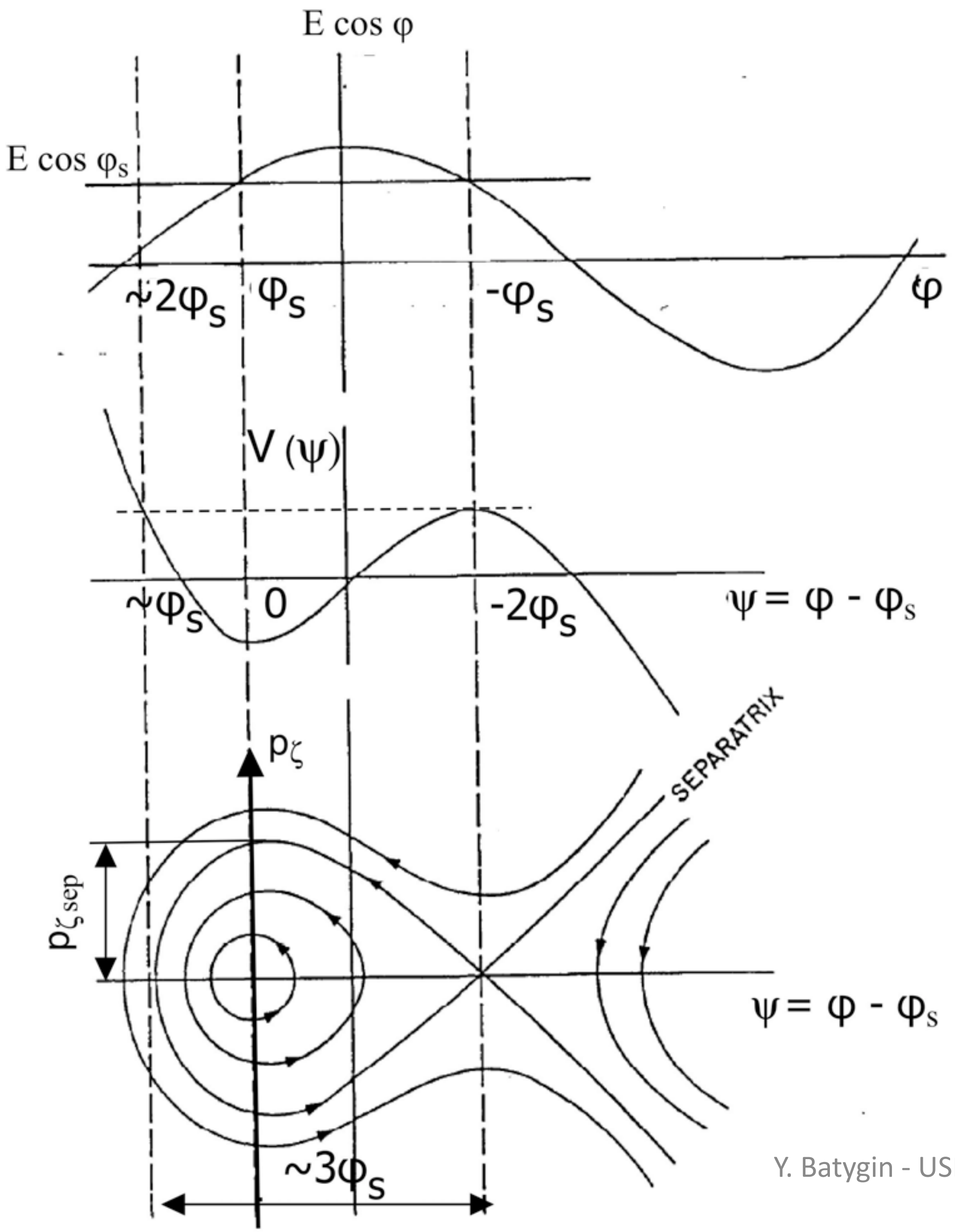
$$H = \frac{(\Delta W)^2}{2m\gamma_s^3 \beta_s^2 c^2}\omega + qE\beta c[\sin(\varphi_s + \psi) - \psi \cos\varphi_s]$$

Equations of motions:

$$\frac{d\Delta W}{dt} = qE\beta c[\cos\varphi_s - \cos(\varphi_s + \psi)]$$

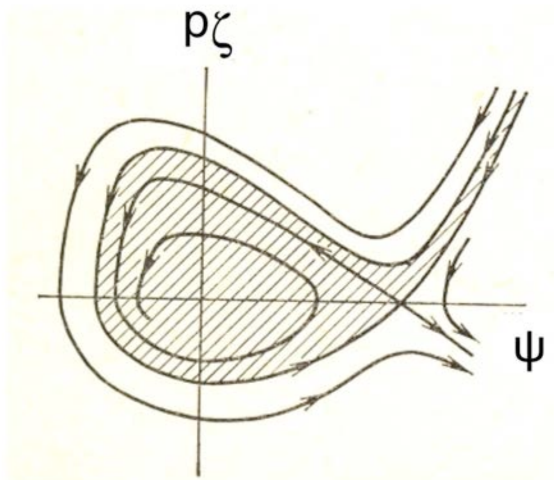
$$\frac{d\psi}{dt} = \frac{\Delta W}{m\gamma_s^3 \beta_s^2 c^2}\omega$$

Accelerating Field, Potential Function, and Separatrix



Potential function:

$$V(\psi) = \frac{qE}{k_z}[\sin(\varphi_s + \psi) - \psi \cos\varphi_s]$$



Separatrix of longitudinal phase space oscillations including acceleration.

Equation of Separatrix

Derivative of potential function determines two extremum points: stable point $\psi = 0$
unstable point $\psi = -2\varphi_s$.

$$\frac{dV}{d\psi} = \frac{qE}{k_z}[\cos(\varphi_s + \psi) - \cos\varphi_s] = 0$$

To be stable, potential function must have minimum in extremum point $\psi = 0$, or the second derivative hast to be positive

$$\frac{d^2V(0)}{d\psi^2} = -\frac{qE}{k_z}\sin\varphi_s > 0$$

Stability condition $\sin\varphi_s < 0$

$$\boxed{\varphi_s < 0}$$

Hamiltonian, corresponding to separatrix

$$H_{sep} = H(p_\zeta = 0, \psi = -2\varphi_s)$$

$$H_{sep} = \frac{qE}{k_z}[-\sin\varphi_s + 2\varphi_s\cos\varphi_s]$$

Equation for separatrix

$$\boxed{\frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s + \psi) - \psi\cos\varphi_s + \sin\varphi_s - 2\varphi_s\cos\varphi_s] = 0}$$

Phase Width of Separatrix

Phase length of separatrix Φ_s is determined from separatrix equation assuming $p_\zeta=0$

$$\sin(\varphi_s+\psi)-\psi\cos\varphi_s+\sin\varphi_s-2\varphi_s\cos\varphi_s=0$$

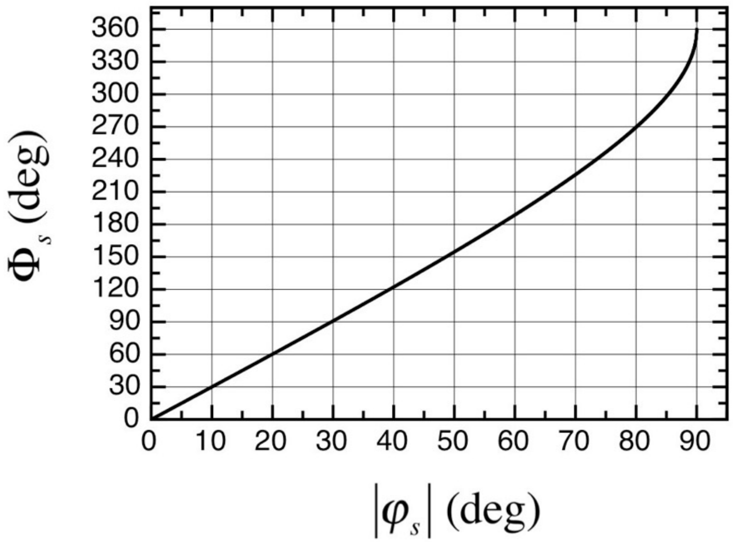
Equation has two roots $\psi_1 = -2\varphi_s$, and ψ_2. Width of separatrix is $\Phi_s = \psi_2 + 2\,|\varphi_s|$
Substitution $\psi_2 = \Phi_s - 2\,|\varphi_s|$ into upper equation gives expression for determination of phase width of separatrix:

$$tg|\varphi_s| = \frac{\Phi_s - \sin\Phi_s}{1-\cos\Phi_s}$$

For small values of synchronous phase, $tg\,\varphi_s \approx \varphi_s$ $\sin\Phi_s \approx \Phi_s - \Phi_s^3/6$
$\cos\Phi_s \approx 1-\Phi_s^2/2$ phase width of separatrix

$$\Phi_s \approx 3|\varphi_s|$$

Therefore, $\psi_2 \approx \varphi_s$



Phase width of separatrix as a function of synchronous phase.

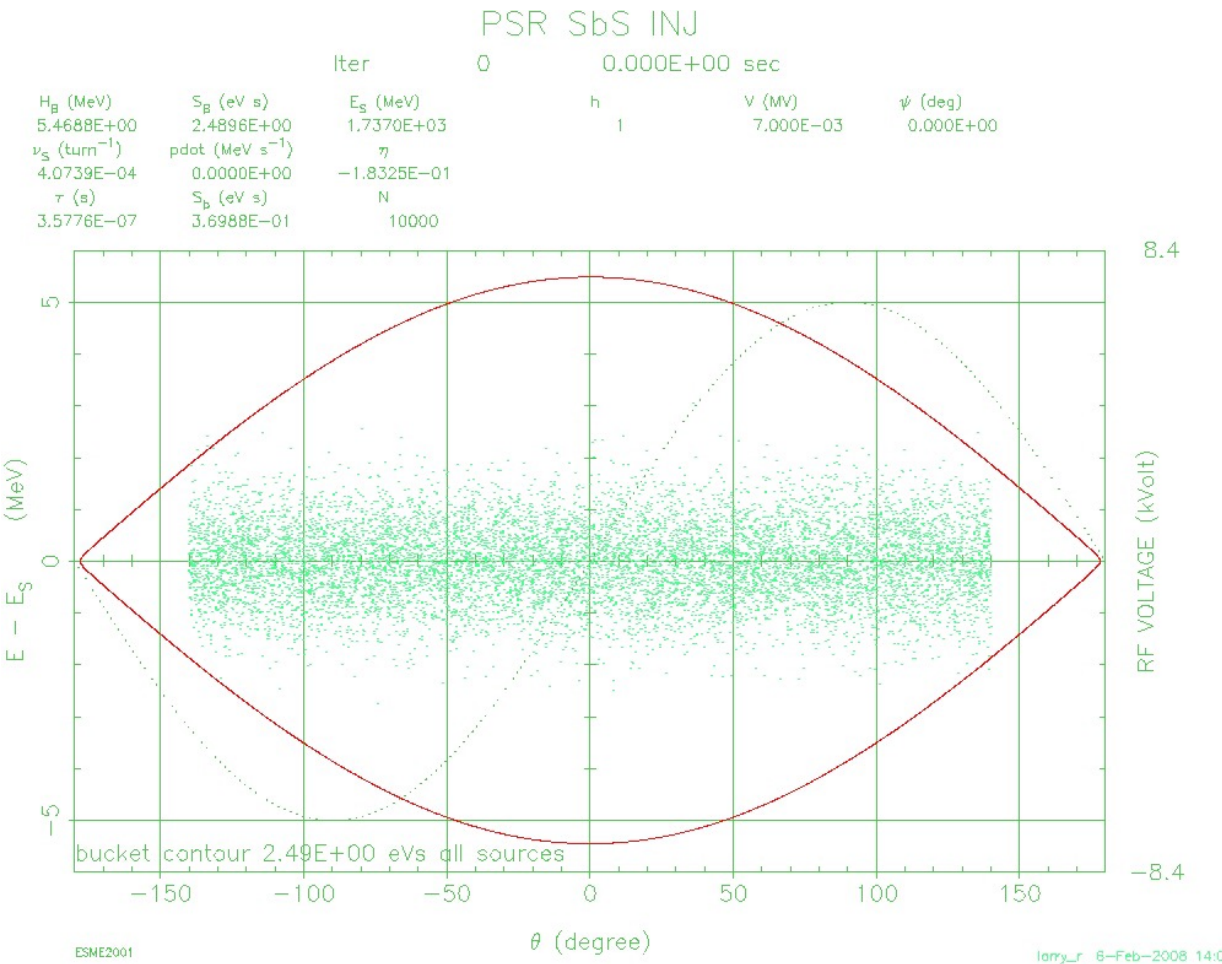
Longitudinal Oscillations Around Synchronous Particle

Equation for longitudinal oscillations around synchronous particle

$$\frac{d^2\zeta}{dt^2} = \frac{qE}{m\gamma^3}[\cos(\varphi_s - k_z\zeta) - \cos\varphi_s]$$

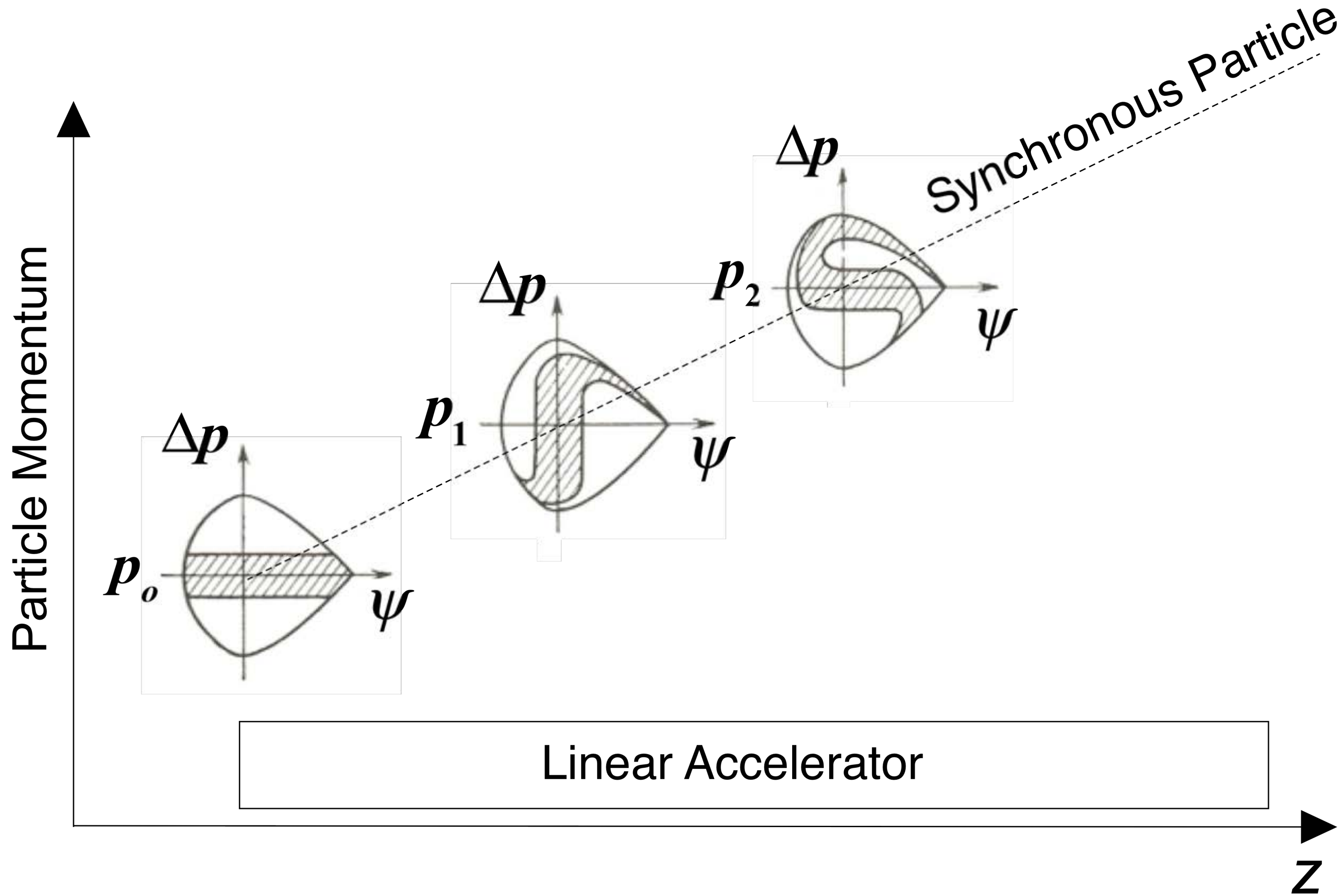
At the separatrix $k_z\zeta = 2\varphi_s$, frequency is zero:

$$\cos(\varphi_s - k_z\zeta) - \cos\varphi_s = 0$$



Longitudinal oscillations in RF field with φ_s =- 90^o (Courtesy of Larry Rybarcyk).

Acceleration and Oscillations Around Synchronous Particle



Frequency of Linear Small Amplitude Oscillations

Equation for longitudinal oscillations

$$\frac{d^2\zeta}{dt^2} = \frac{qE}{m\gamma^3}[\cos(\varphi_s - k_z\zeta) - \cos\varphi_s]$$

For small amplitude oscillations

$$\cos(\varphi_s - k_z\zeta) \approx \cos\varphi_s + k_z\zeta\sin\varphi_s$$

$$\frac{d^2\zeta}{dt^2} + \left(\frac{qE\,k_z\,|\sin\varphi_s|}{m\gamma^3}\right)\zeta = 0$$

Frequency of small amplitude linear oscillations

$$\Omega = \sqrt{\frac{qEk_z|\sin\varphi_s|}{m\gamma^3}}$$

$$\frac{\Omega}{\omega} = \sqrt{\frac{qE\lambda}{mc^2}\frac{|\sin\varphi_s|}{2\pi\beta\gamma^3}}$$

At the separatrix $k_z\zeta=2\varphi_s$, frequency is zero:

$$\cos(\varphi_s - k_z\zeta) - \cos\varphi_s = 0$$

Hamiltonian of Linear Small Amplitude Oscillations

From Hamiltonian of longitudinal oscillations

$$H = \frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s]$$

expanding trigonometric function

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - k_z\zeta\cos\varphi_s - \frac{(k_z\zeta)^2}{2}\sin\varphi_s$$

Hamiltonian of small linear oscillations:

$$\boxed{H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2}}$$

Phase Advance of Longitudinal Oscillations

Equation of linear longitudinal oscillations

$$\frac{d^2\zeta}{dt^2} + \Omega^2 \zeta = 0$$

Change variable $z = \beta c t$

$$\frac{d^2\zeta}{dz^2} + \left(\frac{\Omega}{\beta c}\right)^2 \zeta = 0$$

Solution of equation of longitudinal oscillations

$$\zeta = \zeta_o \cos\left(\frac{\Omega}{\beta c} z + \psi_o\right)$$

Let *S* to be a period of focusing structure. Phase advance of longitudinal oscillations per focusing period

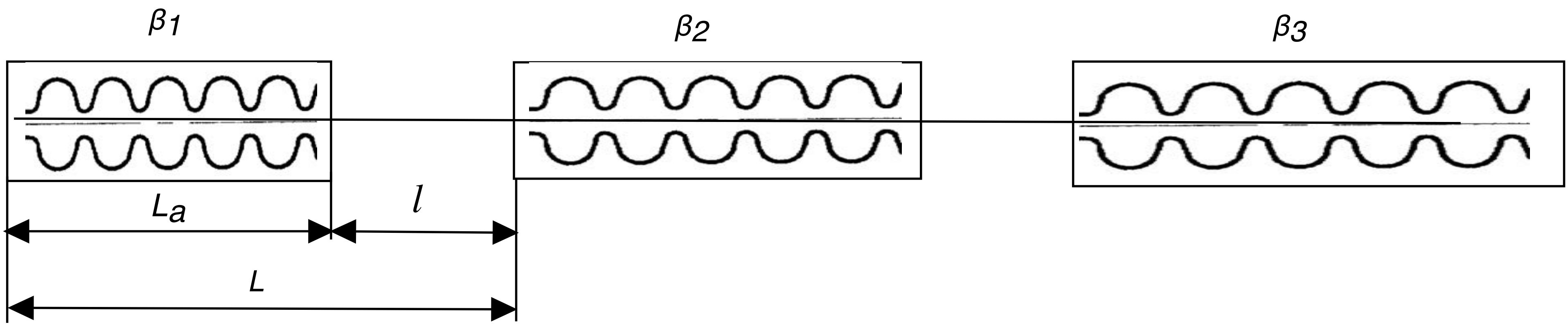
$$\mu_{oz} = \frac{\Omega}{\beta c} S = \sqrt{2\pi \left(\frac{qE\lambda}{mc^2}\right) \frac{|\sin\varphi_s|}{\beta\gamma^3}} \left(\frac{S}{\beta\lambda}\right)$$

Phase advance of longitudinal oscillations per accelerating period

$$\mu_{oa} = \frac{\Omega}{\beta c} L_a = \sqrt{2\pi \left(\frac{qE\lambda}{mc^2}\right) \frac{|\sin\varphi_s|}{\beta\gamma^3}} \left(\frac{L_a}{\beta\lambda}\right)$$

For Alvarez structure $L_a = \beta\lambda$, for π – mode structures $L_a = \beta\lambda/2$

Phase Advance Including Drift Space



Synchronous phase $\varphi_s \approx \varphi_{ref}$

Phase advance of longitudinal oscillations
In single tank

$$\mu_{oa} = \sqrt{2\pi \left(\frac{qE\lambda}{mc^2}\right) \frac{|\sin\varphi_s|}{\beta\gamma^3}} \left(\frac{L_a}{\beta\lambda}\right)$$

Effective accelerating gradient $\tilde{E} = E \frac{L_a}{L_a + l}$

Effective phase advance of longitudinal oscillations per accelerating period *L*

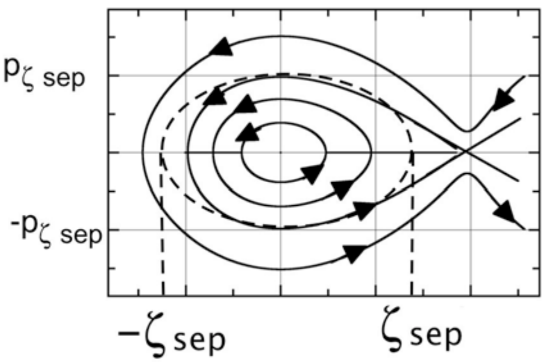
$$\tilde{\mu}_{oa} \approx \sqrt{2\pi \left(\frac{qE\lambda}{mc^2}\right)\left(\frac{L_a}{L_a+l}\right) \frac{|\sin\varphi_s|}{\beta\gamma^3}} \left(\frac{L_a+l}{\beta\lambda}\right) = \mu_{oa}\sqrt{1+\frac{l}{L_a}}$$

Longitudinal Acceptance

Longitudinal acceptance is a phase space area of stable oscillations available for the beam (*area of separatrix*). Let us determine longitudinal acceptance using elliptical approximation to separatrix.

The half - width of separatrix in momentum is determined from separatrix equation assuming $\psi = 0$:

$$\frac{p_{\zeta sep}}{mc} = 2\beta\gamma^3 \frac{\Omega}{\omega}\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}}$$



Elliptical approximation of separatrix

Hamiltonian $$H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2}$$

is constant along elliptical trajectory.

Maximal value of ζ at ellipse is $$m\gamma^3\Omega^2\frac{\zeta_{sep}^2}{2} = \frac{p_{\zeta sep}^2}{2m\gamma^3}$$ or $$\zeta_{sep} = 2\frac{\beta c}{\omega}\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}}$$

Taking approximation $tg\,\varphi_s \approx \varphi_s + \varphi_s^3/3$

$$\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}} \approx \frac{\varphi_s}{\sqrt{3}}$$

Effective length of separatrix

$$\Phi_{s\,eff} = 2\pi\frac{(2\zeta_{sep})}{\beta\lambda} = 4\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}} \approx \frac{4|\varphi_s|}{\sqrt{3}}$$

Longitudinal Acceptance (cont.)

Area of separatrix ellipse is $\pi\,\zeta_{sep}\,(p_{sep}/mc)$. Phase space area of acceptance is determined as a product of ellipse semi-axis:

$$\varepsilon_{acc} = \zeta_{sep}\frac{p_{sep}}{mc} = \frac{2}{\pi}\lambda\beta^2\gamma^3(\frac{\Omega}{\omega})(1-\frac{\varphi_s}{tg\varphi_s})$$

The value of π is not included in the value of acceptance, but is included in units of acceptance ($\pi\, m\, radian$), or, more often ($\pi\ cm\ mrad$).

Using approximation $1-\frac{\varphi_s}{tg\varphi_s} \approx \frac{\varphi_s^2}{3}$, normalized longitudinal acceptance

$$\boxed{\varepsilon_{acc} = \frac{2}{3\pi}\beta^2\gamma^3(\frac{\Omega}{\omega})\,\varphi_s^2\,\lambda}$$

Often longitudinal acceptance and beam emittance are determined in phase plane (φ - φ_s, W-W_s) in units ($\pi\, keV\, deg$).

Relationship between phase and longitudinal coordinate and between energy and momentum

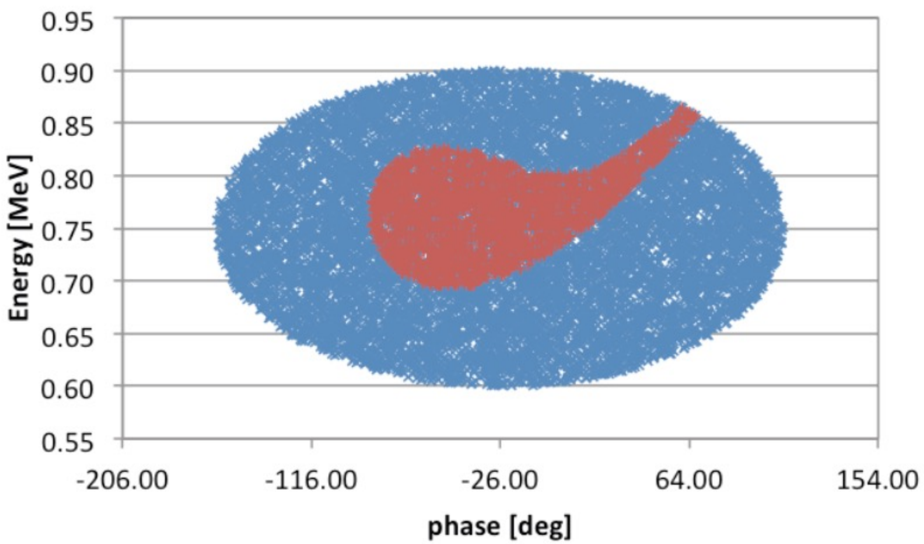
$$\Delta\varphi = 360^o\frac{\zeta}{\beta\lambda}$$

$$\Delta W = mc^2\beta(\frac{\Delta p}{mc})$$

Transformation of longitudinal phase space area in different units:

$$\boxed{\varepsilon_{acc}\ [\pi\cdot keV\cdot \deg] = \varepsilon_{acc}\ [\pi\cdot m\cdot rad]\frac{360^o}{\lambda[m]}\ mc^2[keV]}$$

Longitudinal Acceptance: Example



LANL DTL Acceptance (red)

Accelerating gradient $E=E_oT$	1.6 MV/m
Synchronous phase φ_s	-26°
Wavelength, λ	1.49 m
Energy	750 keV
Velocity, β	0.04

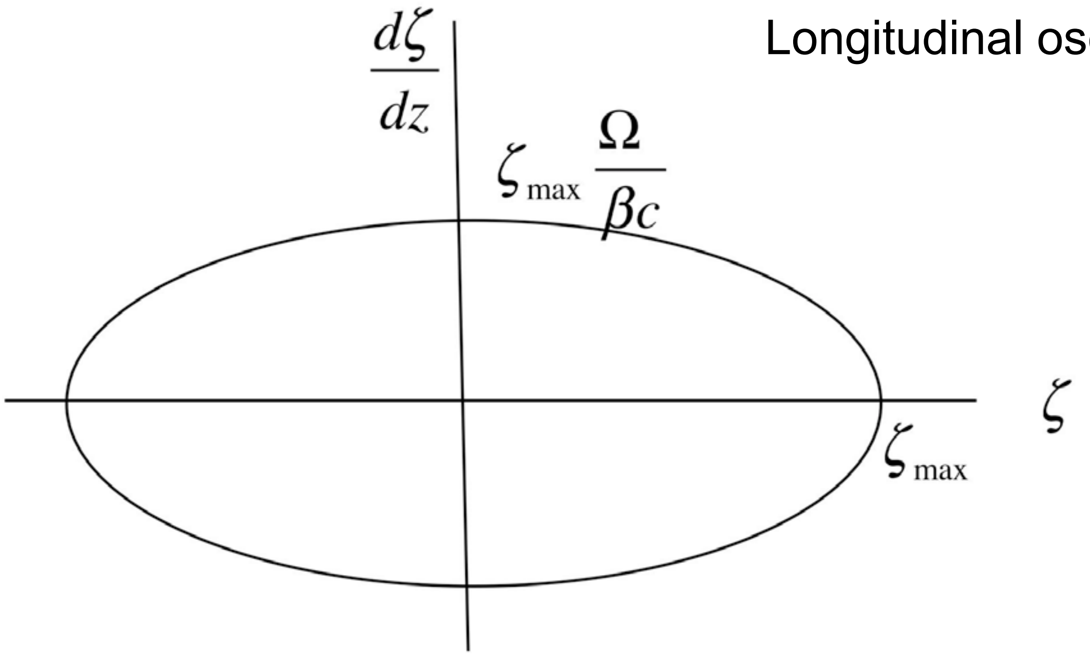
Longitudinal frequency:

$$\frac{\Omega}{\omega}=\sqrt{\frac{qE\lambda}{mc^2}\frac{|\sin\varphi_s|}{2\pi\beta\gamma^3}}=0.0665$$

DTL Longitudinal acceptance:

$$\varepsilon_{acc}=\frac{2}{\pi}\lambda\beta^2\gamma^3(\frac{\Omega}{\omega})(1-\frac{\varphi_s}{tg\varphi_s})=7.17\cdot10^{-6}\pi\, m\, rad=1.62\ \pi\ MeV\deg$$

Unnormalized Longitudinal Matched Beam Emittance



Longitudinal oscillations:

$$\frac{d^2\zeta}{dz^2} + (\frac{\Omega}{\beta c})^2\zeta = 0$$

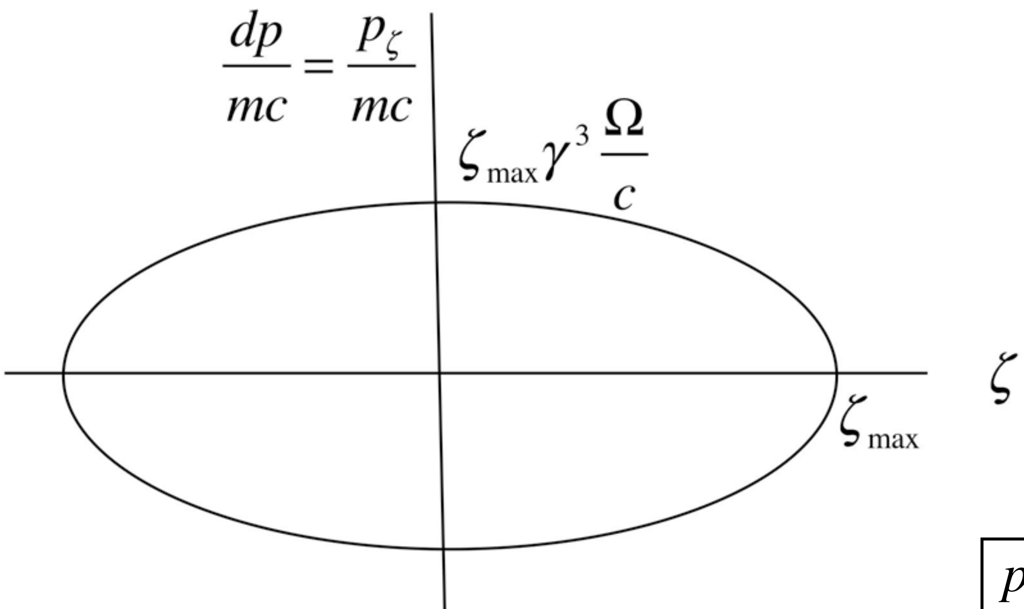
$$\zeta = \zeta_{\max}\cos(\frac{\Omega}{\beta c}z + \psi_o)$$

$$\frac{d\zeta}{dz} = -\zeta_{\max}\frac{\Omega}{\beta c}\sin(\frac{\Omega}{\beta c}z + \psi_o)$$

Unnormalized longitudinal emittance of matched beam:

$$\ni_z = \zeta^2_{\max}\frac{\Omega}{\beta c}$$

Normalized Longitudinal Matched Beam Emittance



$$\frac{d\zeta}{dt} = \Delta(\beta c) \rightarrow d(\beta c)$$

$$d\beta = \frac{1}{\gamma^3}\frac{dp}{mc}$$

$$\frac{d\zeta}{dz} = \frac{d(\beta c)}{\beta c} = \frac{1}{\beta\gamma^3}\frac{dp}{mc}$$

$$\boxed{\frac{p_{\zeta\, max}}{mc} = \zeta_{max}\gamma^3\frac{\Omega}{c} = 2\pi\left(\frac{\zeta_{max}}{\lambda}\right)\left(\frac{\Omega}{\omega}\right)\gamma^3}$$

Normalized longitudinal emittance of matched beam:

$$\boxed{\varepsilon_z = \beta\gamma^3 \ni_z = \zeta^2_{max}\gamma^3\frac{\Omega}{c} = 2\pi\left(\frac{\zeta^2_{max}}{\lambda}\right)\left(\frac{\Omega}{\omega}\right)\gamma^3}$$

Adiabatic Damping of Longitudinal Oscillations

Previous analysis was performed in conservative approximation assuming accelerator parameters are constant along the machine. Consider now effect of acceleration on longitudinal oscillations.

Hamiltonian of linear oscillations

$$H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2}$$

Along phase space trajectory $H = const$. Let us divide expression for Hamiltonian by H. Phase space trajectory is an ellipse

$$\frac{p_\zeta^2}{p_{\zeta\,\max}^2} + \frac{\zeta^2}{\zeta_{\max}^2} = 1$$

Semi-axis of ellipse

$$p_{\zeta\,\max} = \sqrt{2Hm\gamma^3} \qquad \zeta_{\max} = \frac{1}{\Omega}\sqrt{\frac{2H}{m\gamma^3}}$$

The value of Hamiltonian, *H*, is the energy of particle oscillation around synchronous particle. Product of semi-axis of ellipse, gives the value of phase space area comprised by a particle performing linear longitudinal oscillations. Largest phase space trajectory comprises longitudinal beam emittance:

$$\varepsilon_z = \frac{p_{\zeta\,\max}}{mc}\zeta_{\max} = \frac{2H}{mc\Omega}$$

Adiabatic Damping of Longitudinal Oscillations (cont.)

If parameters of accelerator are changing adiabatically along the channel, the value of beam ellipse in phase space is conserved according to theorem of adiabatic invariant. In this case, energy of particle oscillation around synchronous particle, H , is proportional to frequency of longitudinal oscillation, Ω:

$$H \sim \Omega$$

Adiabatic change of parameters means that parameters are changing slowly during one oscillation period of $2\pi/\Omega$.

The semi-axes of beam ellipse are changing as

$$\zeta_{\max} = \sqrt{\frac{\varepsilon_z c}{\gamma^3 \Omega}} \sim \frac{1}{\gamma^{3/2} \Omega^{1/2}} \qquad \frac{p_{\zeta \max}}{mc} = \sqrt{\frac{\gamma^3 \varepsilon_z \Omega}{c}} \sim \gamma^{3/2} \Omega^{1/2}$$

Adiabatic Damping of Longitudinal Oscillations (cont.)

Many accelerators are designed keeping the constant values of equivalent traveling wave, E, and synchronous phase φ_s. In this case, longitudinal oscillation frequency drops as

$$\Omega \sim \frac{1}{\beta^{1/2}\gamma^{3/2}}$$

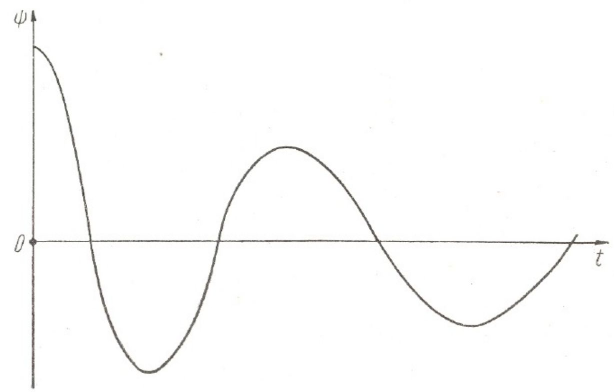
Semi-axes of beam ellipse at phase plane are changing as

$$\zeta_{\max} \sim \frac{\beta^{1/4}}{\gamma^{3/4}} \qquad p_{\zeta\ \max} \sim \frac{\gamma^{3/4}}{\beta^{1/4}}$$

Phase length of the bunch and relative momentum spread drop as

$$\Delta\psi \sim \frac{1}{(\beta\gamma)^{3/4}}$$

$$\frac{\Delta p}{p_s} \sim \frac{1}{\beta^{5/4}\gamma^{1/4}}$$



Adiabatic Phase Damping

Longitudinal Beam Phase Space $\Delta W \Delta \phi = \text{Constant}$

Beam Energy Spread $\Delta W = \text{Constant x } (\beta\gamma)^{3/4}$

Beam Phase Width $\Delta \phi = \dfrac{\text{Constant}}{(\beta\gamma)^{3/4}}$

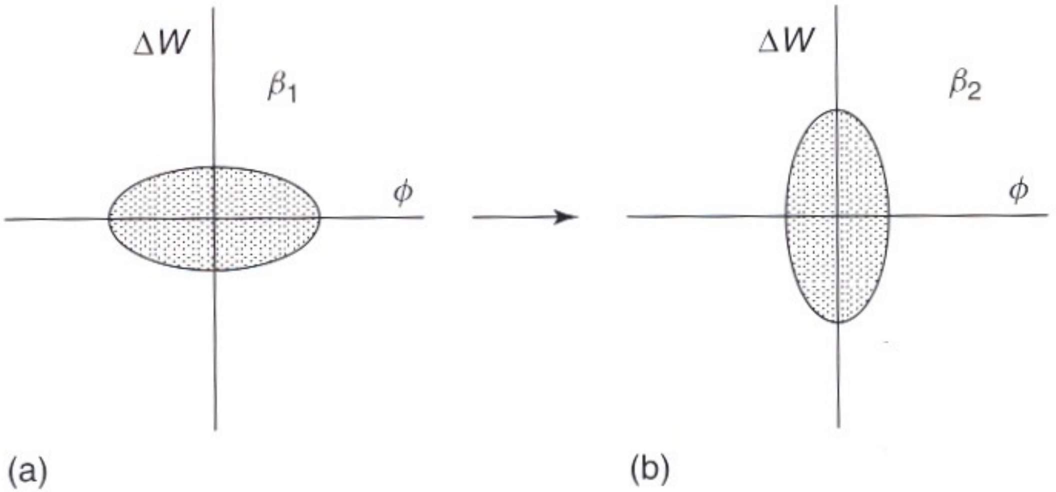
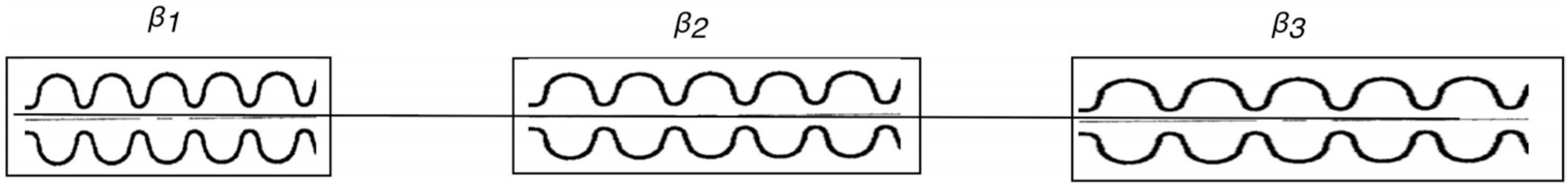


Figure 6.8 Phase damping of a longitudinal beam ellipse caused by acceleration. The phase width of the beam decreases and the energy width increases, while the total area remains constant.

Acceleration in Sections with Constant β

LANSCE high-energy linear accelerator.



Sequence of accelerating structures with constant geometrical velocities $\beta_3 > \beta_2 > \beta_1$.

Acceleration in Sections with Constant β (cont.)

Since in cavities $d\beta_g / dz = 0$, the synchronous phase in every cavity is $\varphi_s = -90^o$. Acceleration is achieved as a rotation in phase space around synchronous phase $\varphi_s = -90^o$ at the finite length of cavity.

Energy-phase equations in a traveling wave with constant phase velocity β_g

$$\frac{d\varphi}{dz} = \frac{2\pi}{\lambda}\left(\frac{1}{\beta} - \frac{1}{\beta_g}\right) \qquad d\varphi / dz = \partial H / \partial \gamma$$

$$\frac{d\gamma}{dz} = \frac{qE}{mc^2}\cos\varphi \qquad d\gamma / dz = -\partial H / \partial \varphi$$

The Hamiltonian of particle motion in the field of equidistance cells is

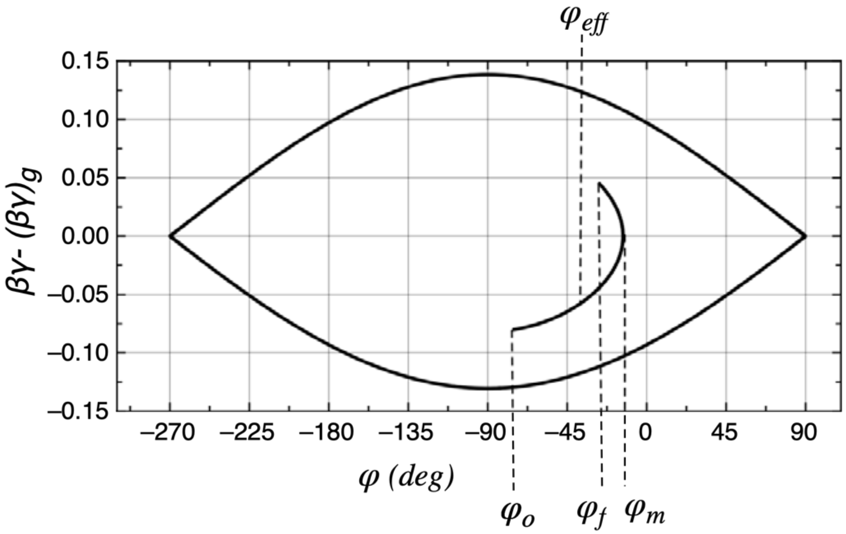
$$H = \frac{2\pi}{\lambda}\left(\beta\gamma - \frac{\gamma}{\beta_g}\right) - \frac{qE}{mc^2}\sin\varphi$$

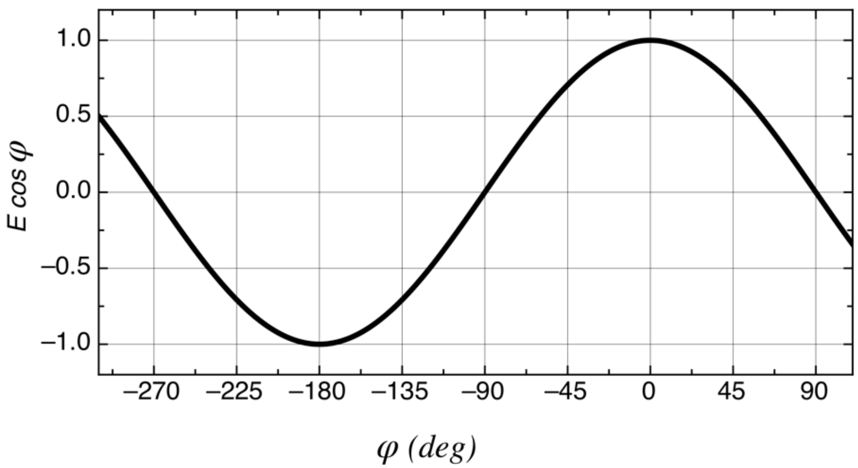
Since β_g = const, $\Delta\beta/\beta \ll 1$, and $E = E_o T(\beta) \sim$ const, the Hamiltonian is a constant of motion.

From Hamiltonian, the beam energy in the structure, γ, as a function of the RF phase of particle, φ, is given by

$$\sqrt{\gamma^2 - 1} - \frac{\gamma}{\beta_g} = \sqrt{\gamma_o^2 - 1} - \frac{\gamma_o}{\beta_g} + \frac{qE\lambda}{2\pi mc^2}(\sin\varphi - \sin\varphi_o)$$

Acceleration in Sections with Constant β (cont.)





(Up) phase space trajectory of a particle in an RF structure with equidistant cells, (bottom) equivalent traveling wave.

Final energy $\gamma_f(\varphi_f)$ versus initial energy γ_o and initial phase φ_o :

$$\gamma_f = \gamma_g \pm \sqrt{\frac{qE\lambda(\beta_g\gamma_g)^3}{\pi mc^2}} \cdot \sqrt{\sin\varphi_o - \sin\varphi_f + \frac{\pi}{(\beta_g\gamma_g)^3}\frac{mc^2}{qE\lambda}(\gamma_g-\gamma_o)^2}$$

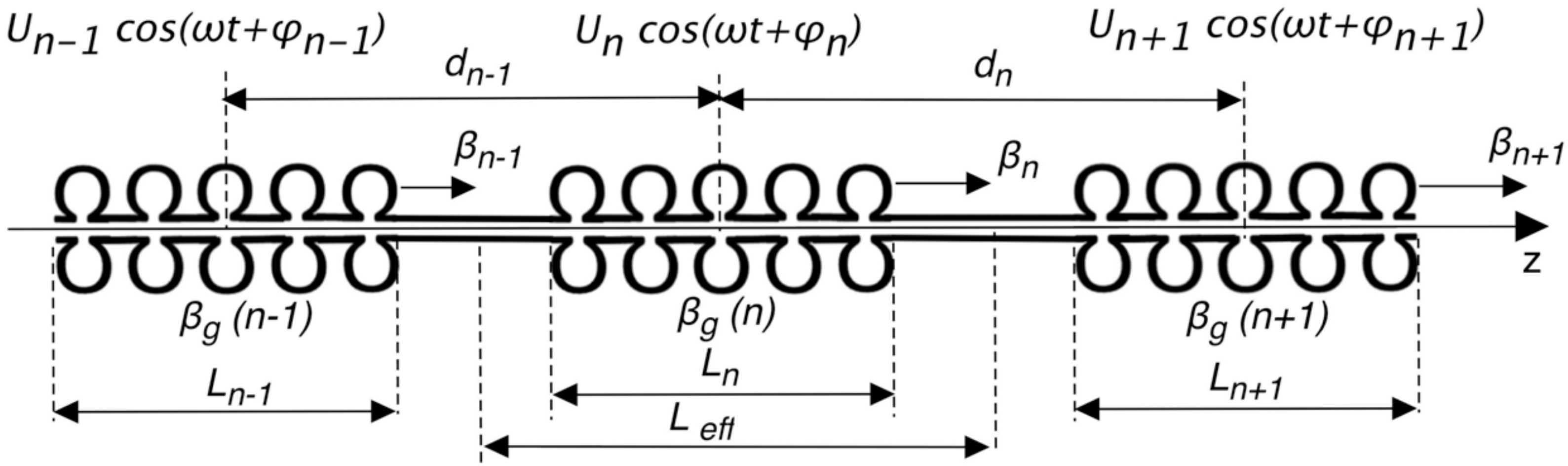
Dimensionless time of particle acceleration in the cavity, $(\Delta\omega t)$, versus phase slippage from φ_o to φ_f

$$\Delta(\omega t) \approx \sqrt{\frac{2\pi\beta_g\gamma_g^3 mc^2}{qE\lambda|\sin\varphi_m|}} \cdot \{\arcsin[1+(\varphi_m-\varphi_f)\tan\varphi_m] - \arcsin[1+(\varphi_m-\varphi_o)\tan\varphi_m]\}$$

Number of accelerating cells: $N_{cell} = \dfrac{\Delta\omega t}{\pi}$

Reference: Y.B., NIM-A, 1040, (2022) 167192.

Acceleration in Sections with Constant β (cont.)



Accelerating structure of independently phased cavities.

In an array of accelerating cavities with constant β_g the acceleration is achieved with the shifts of RF phases between cavities.

The velocity of the reference particle after cavity (n):

$$\beta_n = \frac{2\pi d_n}{\lambda(\varphi_n - \varphi_{n+1})}$$

The synchronous phase of accelerating wave along linac

$$\cos\varphi_s(z) = \frac{mc^2}{q\bar{E}}\beta_n \gamma_n^3 \frac{d\beta_n}{dz}$$

Amplitude of accelerating wave

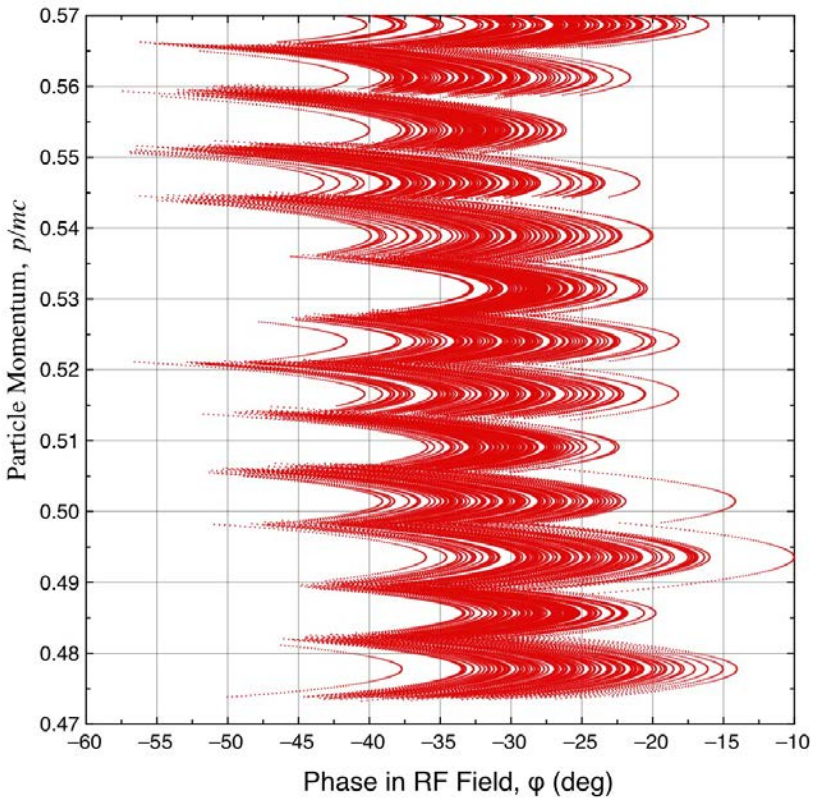
$$\bar{E} = E_{o\_n}\bar{T}_n(\beta_{s\_n})\frac{L_n}{L_{eff}}$$

Velocity of the reference particle within cavity (n)

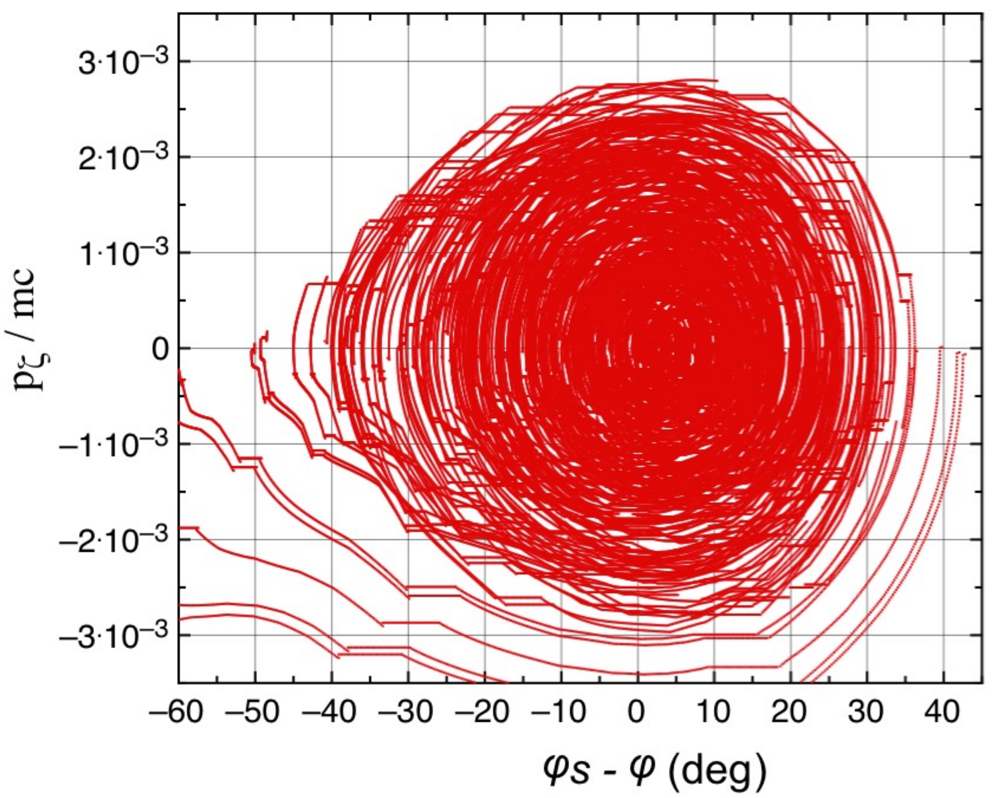
$$\beta_{s\_n} = \frac{\beta_{n-1}+\beta_n}{2}$$

Acceleration in LANSCE High-Energy Linac

Dynamics in RF field of multiple sections with constant β



Dynamics around synchronous particle



Dynamics in Sections with $\beta_g = 1$

In accelerating sections with β_g = 1 there is no synchronous particle.

Energy-phase equations in a traveling wave with geometrical velocity β_g = 1

$$\frac{d\varphi}{dz} = \frac{2\pi}{\lambda}\left(\frac{1}{\beta} - 1\right)$$

$$\frac{d\gamma}{dz} = \frac{qE}{mc^2}\cos\varphi$$

Hamiltonian of particle motion in a section with geometrical velocity β_g = 1

$$H = \frac{2\pi}{\lambda}(\beta\gamma - \gamma) - \frac{qE}{mc^2}\sin\varphi$$

From Hamiltonian, the constant of motion is

$$C = \gamma - \sqrt{\gamma^2 - 1} + \frac{qE\lambda}{2\pi mc^2}\sin\varphi$$

From the constant of motion, the energy of the particle $\gamma(\varphi)$ as a function of initial energy and phase is

$$\gamma(\varphi) = \frac{1}{2}\left[\gamma_o - \sqrt{\gamma_o^2 - 1} + \left(\frac{qE\lambda}{2\pi mc^2}\right)(\sin\varphi_o - \sin\varphi) + \frac{1}{\gamma_o - \sqrt{\gamma_o^2 - 1} + \left(\frac{qE\lambda}{2\pi mc^2}\right)(\sin\varphi_o - \sin\varphi)}\right]$$

Dynamics in Sections with $\beta_s = 1$ (cont.)

For particles with β --> 1 the value of constant

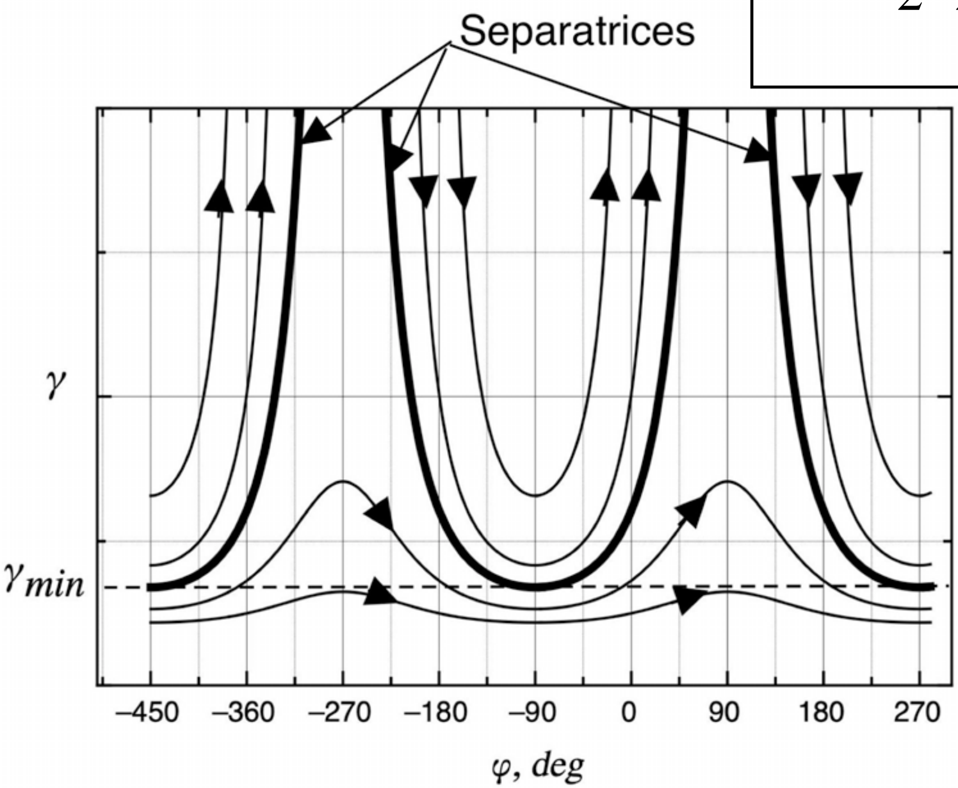
$$C = \frac{qE\lambda}{2\pi mc^2}\sin\varphi_\infty$$

For particles captured in acceleration, the value of the C is limited

$$C \le \frac{qE\lambda}{2\pi mc^2}$$

The energy of injected particles to be captured into continuous unlimited acceleration has to be $\gamma_o \ge \gamma_{sep}$

$$\gamma_{sep} = \frac{1}{2}\left[\frac{qE\lambda}{2\pi mc^2}(1-\sin\varphi_o) + \frac{1}{\left(\frac{qE\lambda}{2\pi mc^2}\right)(1-\sin\varphi_o)}\right]$$



Longitudinal particle trajectories in the field with $\beta_g = 1$.

Minimal Energy of Particles Accelerated in Wave with $\beta_g = 1$

Only particles whose initial energy $\gamma_o(\varphi) > \gamma_{sep}(\varphi)$ can be captured into continuous unlimited acceleration

The equation $\boxed{\partial\gamma_{sep} / \partial\varphi = 0}$ determines phase $\varphi = -\pi/2$ where the initial energy has the minimal possible value:

$$\boxed{\gamma_{\min} = \frac{1 + \left(\frac{qE\lambda}{\pi mc^2}\right)^2}{2\frac{qE\lambda}{\pi mc^2}}}$$

For proton in linac with E = 5 MeV/m, λ = 1m, the value of γ_{min} = 294.76, or W_{min}= <u>275 GeV</u>.

However, ion beams can be accelerated in finite-length sections with β_g = 1 within phase area $-\pi/2 < \varphi < \pi/2$.

RF Cavities Tuning: Threshold Field

The increment of energy that the equilibrium particle receives during each acceleration period is determined by the increase in the period length and, therefore, is determined by the design od accelerator:

$$\Delta W_s = eE_o TL \cos\varphi_s = const$$

The threshold field at which the equilibrium phase is still real ($\cos \varphi_s = 1$) is

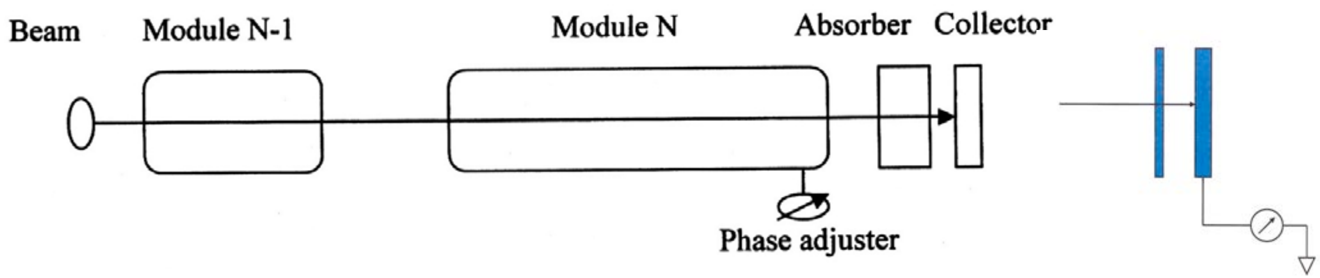
$$E_{th} = \frac{\Delta W_s}{eTL}$$

Accelerating field must be $E_o \geq E_{th}$

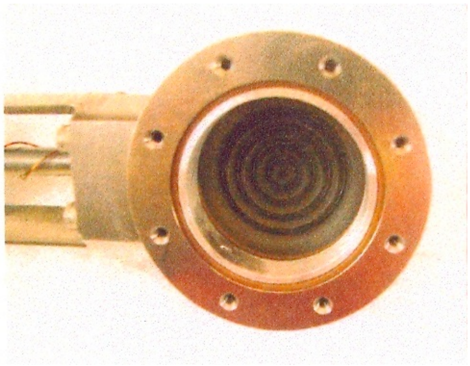
Synchronous phase is $\cos\varphi_s = \frac{E_{th}}{E_o}$

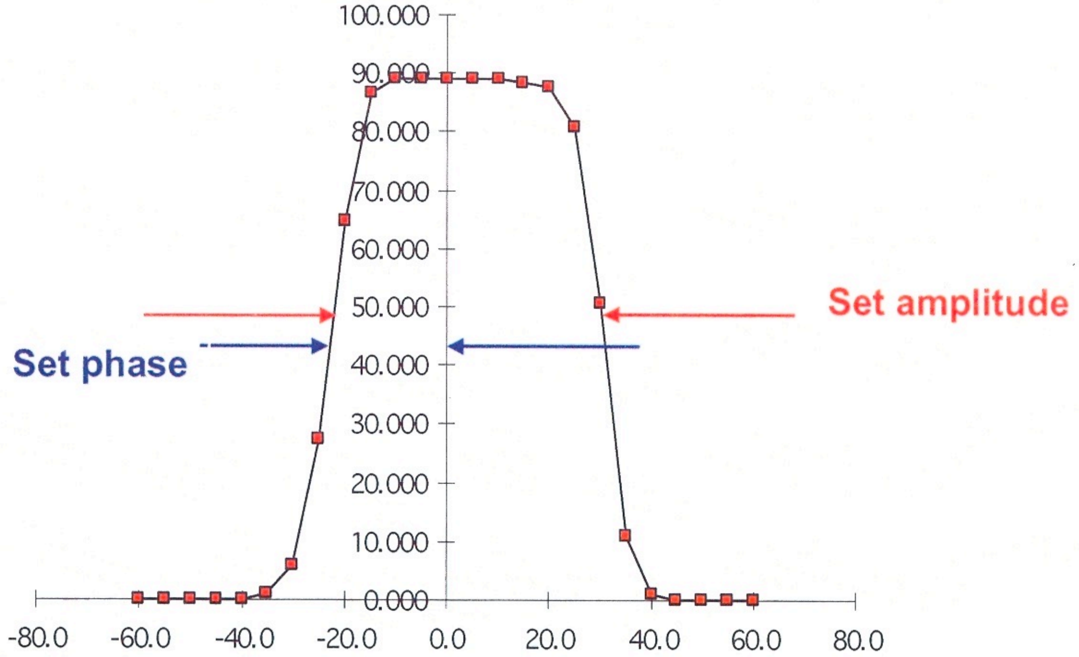
The threshold field is determined through measurement of width of energy capture region as a function of field in resonator. This is done by measurement of dependence of accelerated beam current versus injection energy. The threshold field is determined by extrapolating of the energy width of capture region to zero value.

Phase Scans to Set the Phase and Amplitude of RF Linac



Schematic of the phase scan measurement setup. At LANL linac there are 4 absorber/collectors at 40, 70, 100, and 121 MeV.





Result of phase scan

Phase Scans to Set the Phase and Amplitude of RF Linac

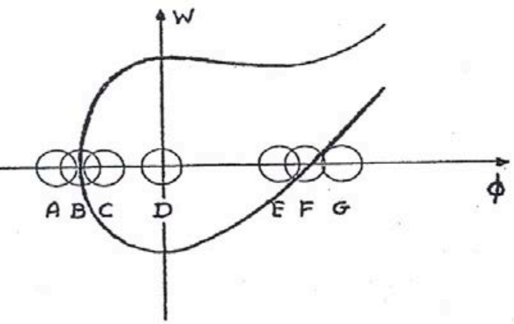
Energy gain of synchronous particle per gap is constant

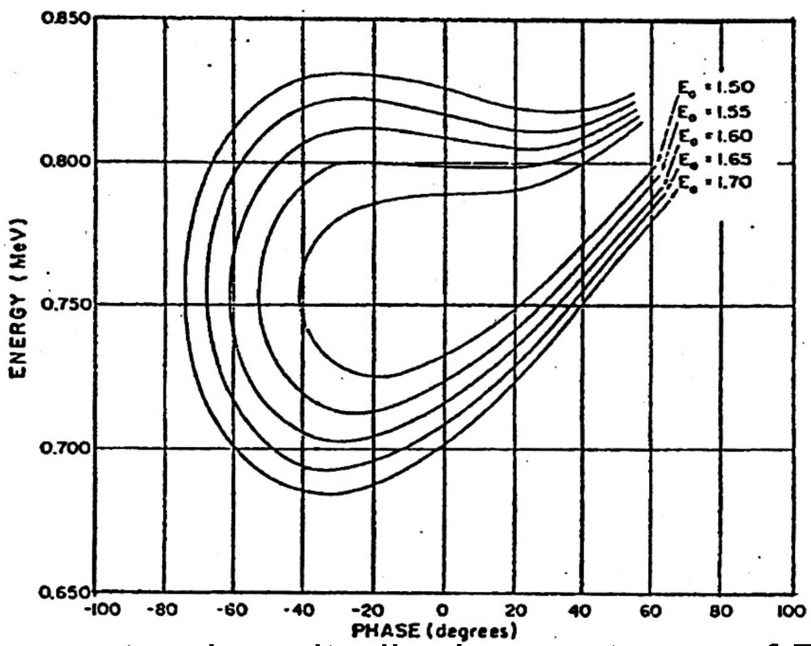
$$\Delta W_s = eEL\cos\varphi_s = const$$

Decrease of accelerating field results in decrease of phase width of separatirx (and vise versa)

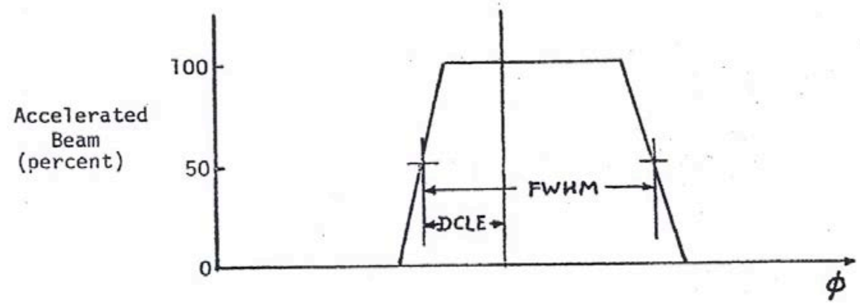
$$E\downarrow \rightarrow \cos\varphi_s \uparrow \rightarrow \varphi_s \downarrow \rightarrow \Phi_{sep} \approx 3\varphi_s \downarrow$$

$$E\uparrow \rightarrow \cos\varphi_s \downarrow \rightarrow \varphi_s \uparrow \rightarrow \Phi_{sep} \approx 3\varphi_s \uparrow$$

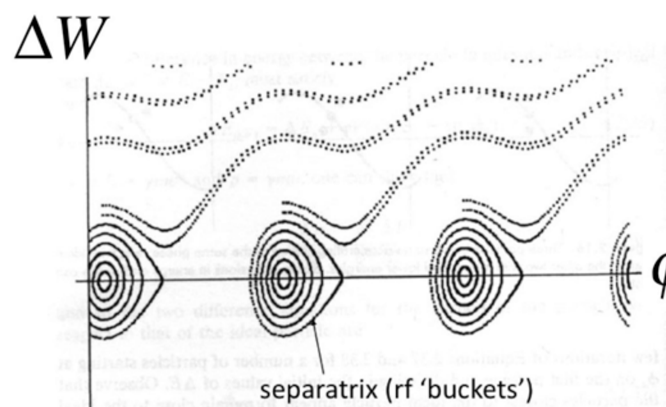




Longitudinal acceptance of RF linac for 5 different average axial field amplitudes.

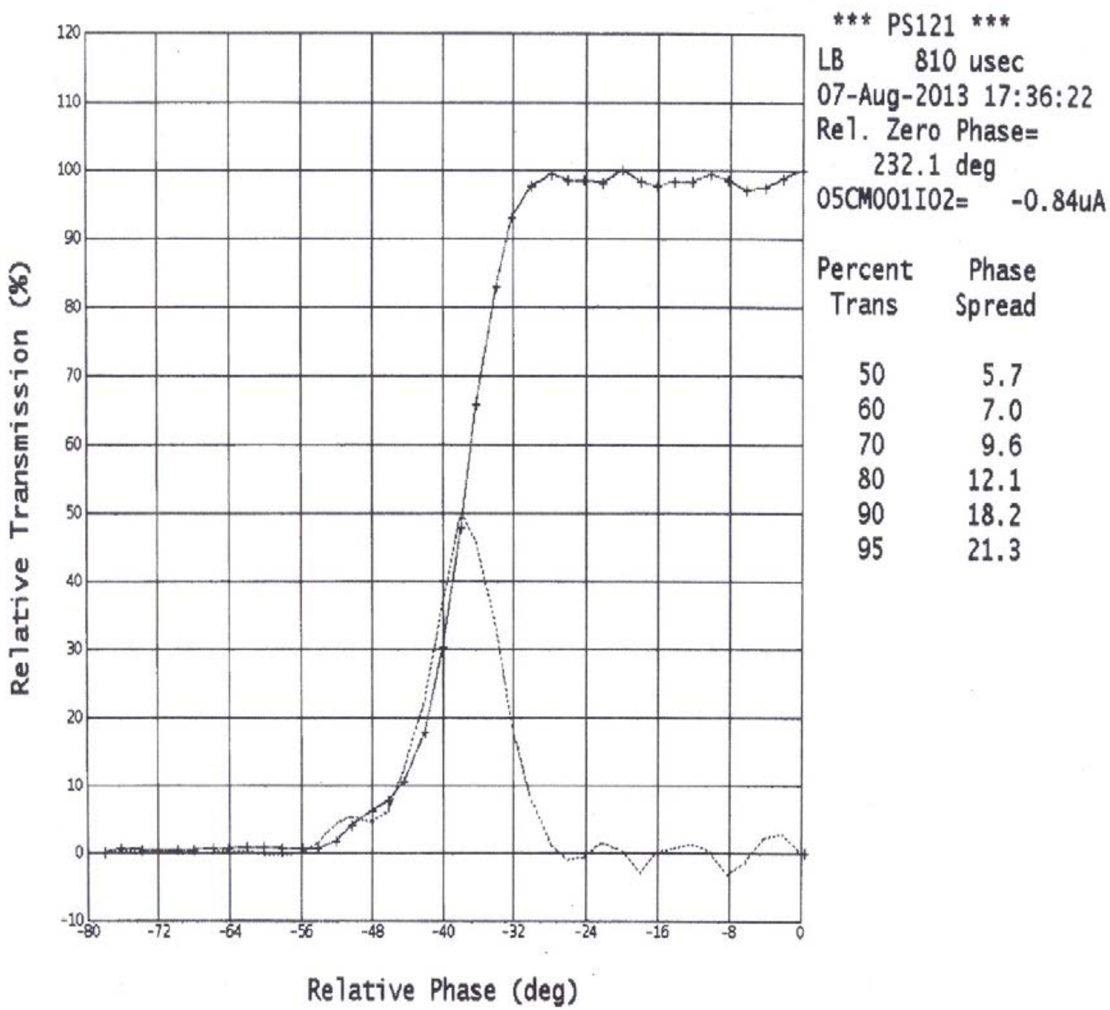


Accelerated beam as a function of beam phase



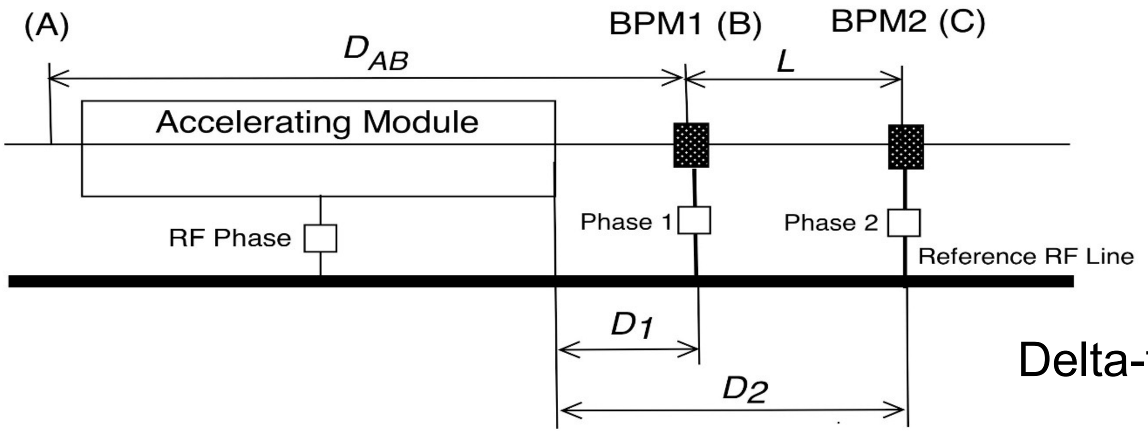
Sequence of RF buckets

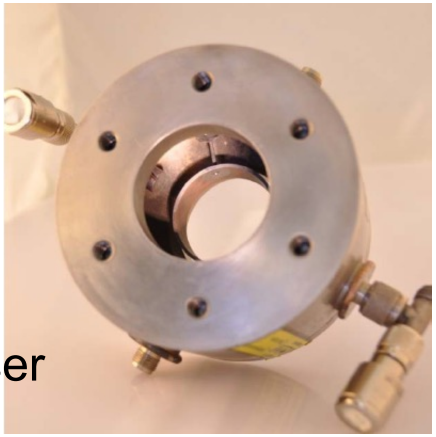
Determination of Bunch Length Using Phase Scan



LANL Phase Scan at the energy of 121 MeV.

Delta-t Tuning Procedure





Delta-t tranduser

Time-of-flight of the beam centroid from location A to B and from A to C: t_{AB}, t_{AC}

Change in $t_{AB,}$ t_{AC} values when accelerating module is switched from off to on are

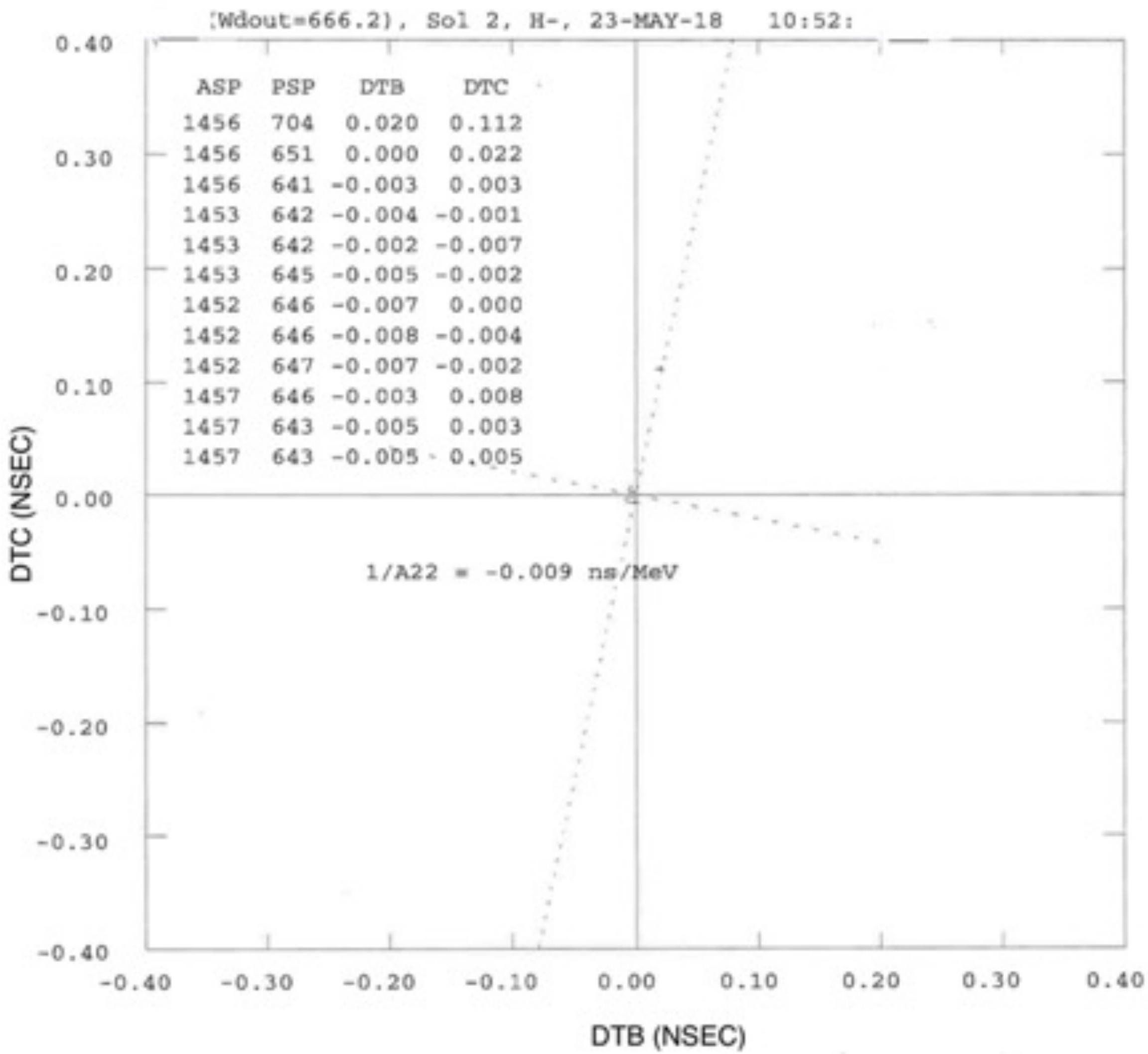
$$t_B = t_{AB,off} - t_{AB,on} \qquad t_C = t_{AC,off} - t_{AC,on}$$

Deviation of values $t_{B,}$ t_C from design values:

$$\Delta t_B = -\frac{D_{AB}}{mc^3(\beta\gamma)_A^3}\Delta W_A - \frac{\Delta\varphi_B - \Delta\varphi_A}{\omega} - \frac{D_1}{mc^3}\left[\frac{\Delta W_A}{(\beta\gamma)_A^3} - \frac{\Delta W_B}{(\beta\gamma)_B^3}\right]$$

$$\Delta t_C = \Delta t_B - \frac{D_2 - D_1}{mc^3}\left[\frac{\Delta W_A}{(\beta\gamma)_A^3} - \frac{\Delta W_B}{(\beta\gamma)_B^3}\right]$$

Delta-t Tune



Output of delta-t program displaying search of amplitude (ASP) and phase (PSP) while minimizing values of (DTB) and (DTC).

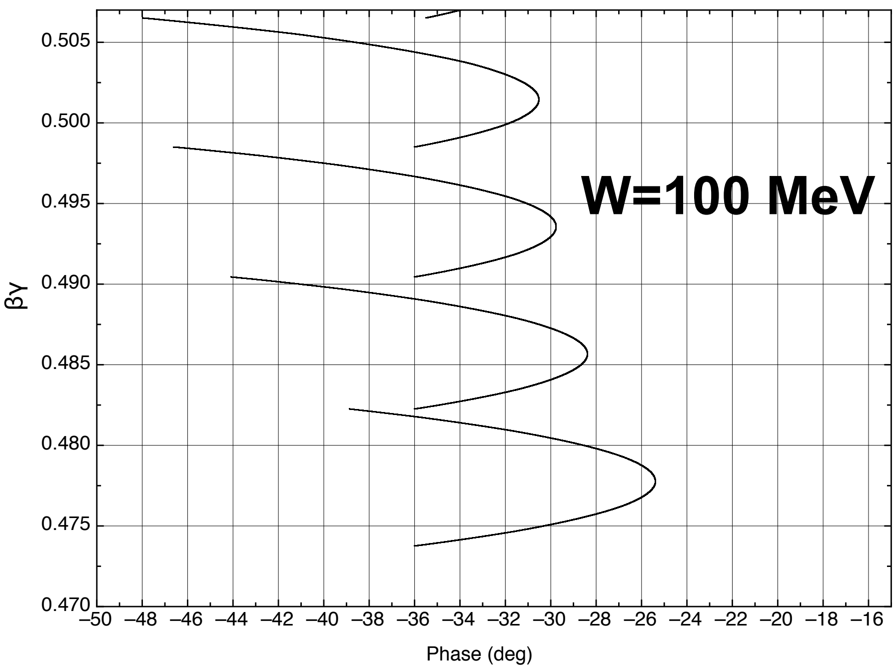
Delta-t Tuning Issues

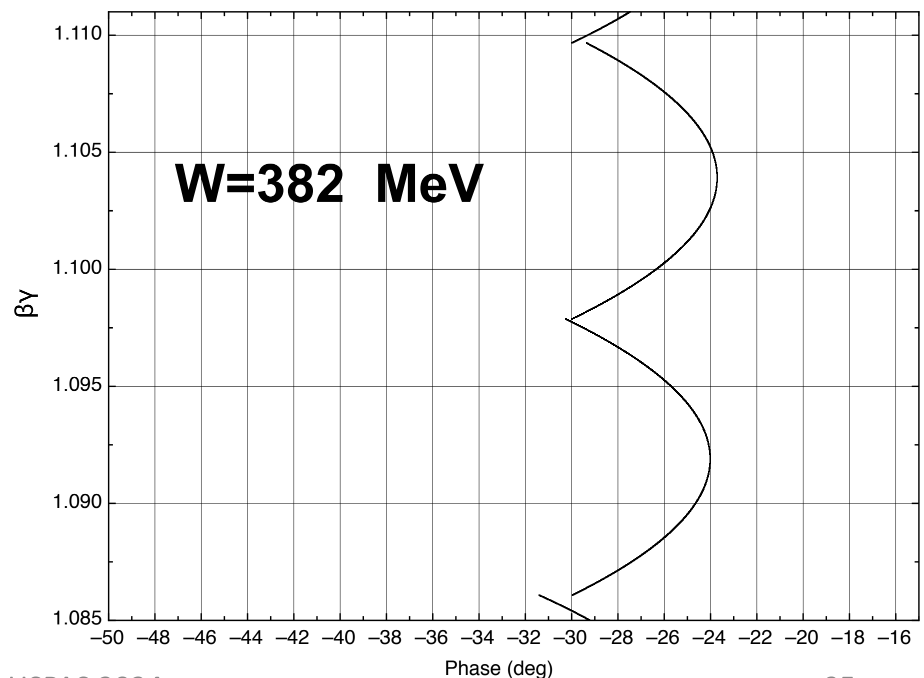
Delta-t tuning procedure works well only when particles perform significant longitudinal oscillations within RF tanks. If longitudinal oscillations are "frozen", then combination of Δt_B, Δt_c can be obtained with infinitely large number of combinations of (E, φ_s).

In linac phase advance of longitudinal oscillation per module drops as

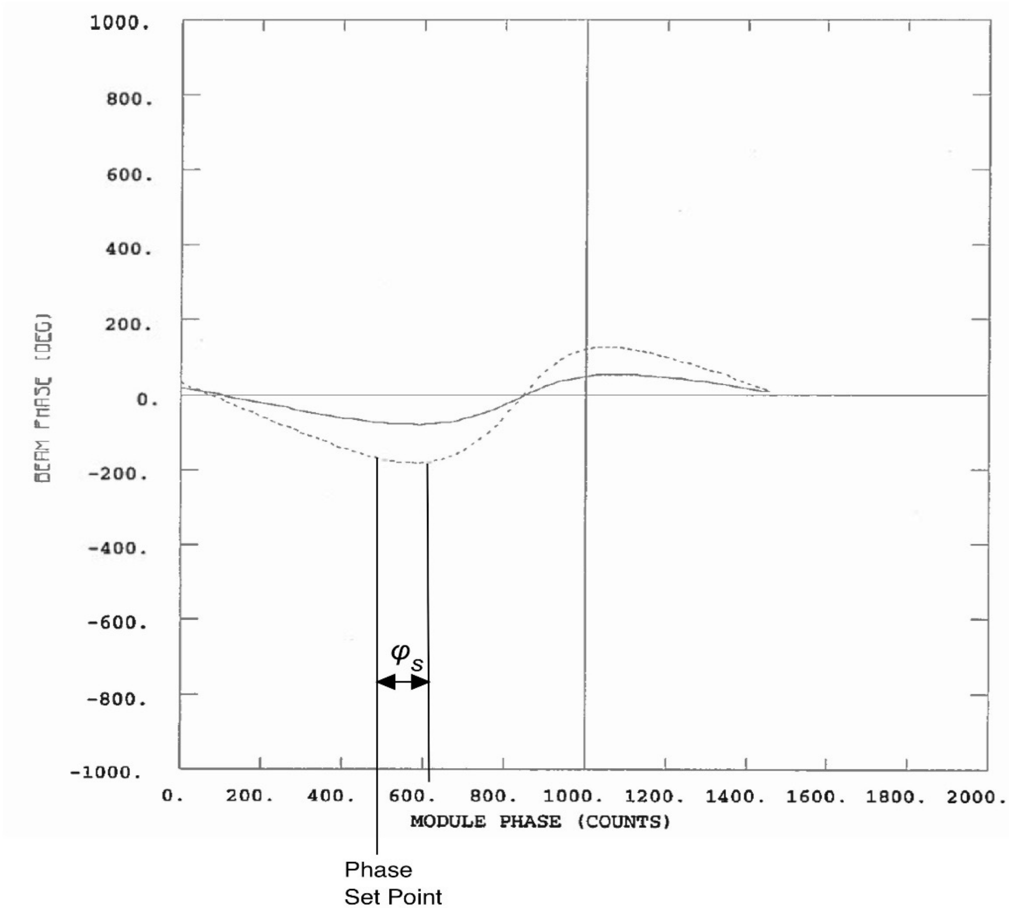
$$\mu_{oa} \sim \sqrt{\frac{E\left|\sin\varphi_s\right|}{(\beta\gamma)^3}}$$

Phase Oscillations



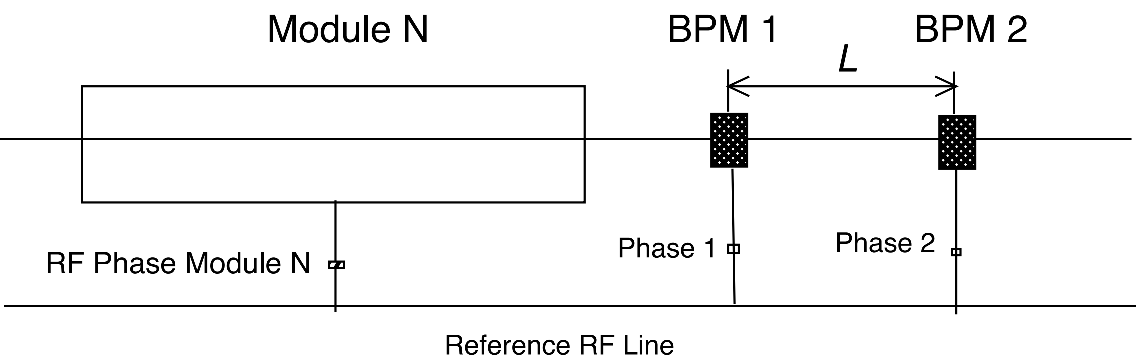


Phase Scans



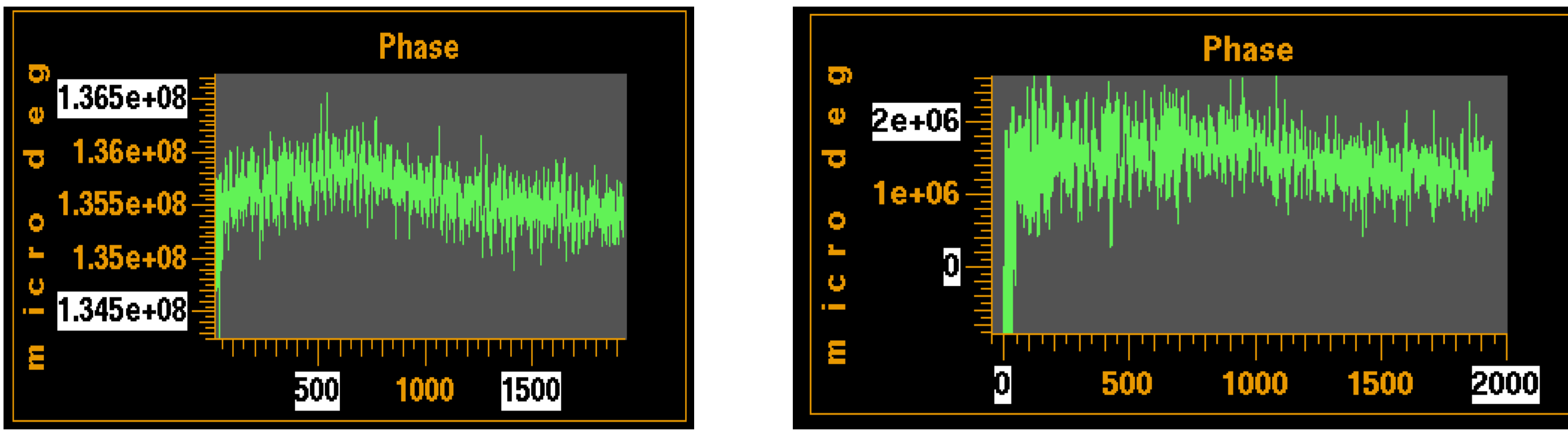
Phase scan: measurement of time of arrival of the beam to downstream pickup loop versus RF phase of the accelerating module.

Measurement of Beam Energy by Difference in BPM RF Phases



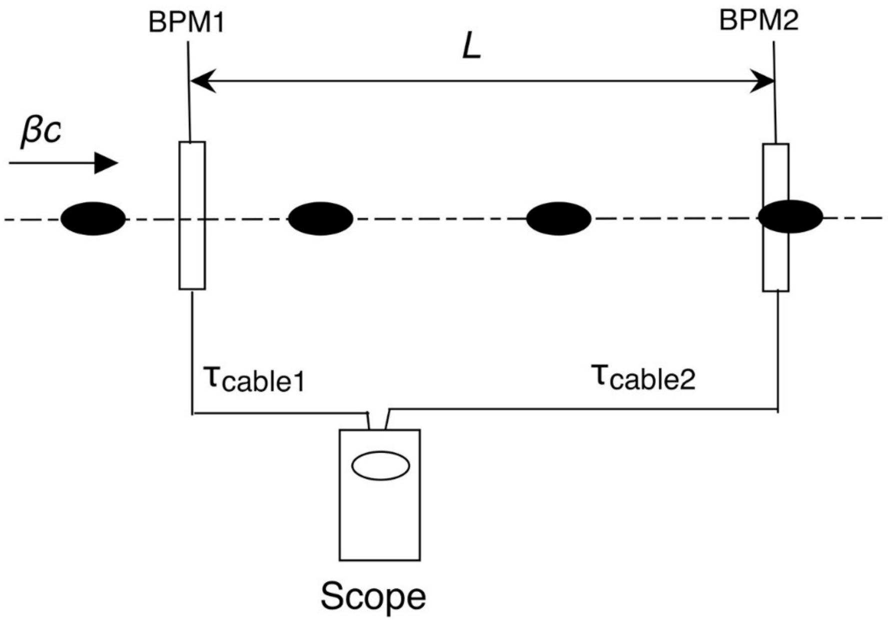
Beam velocity

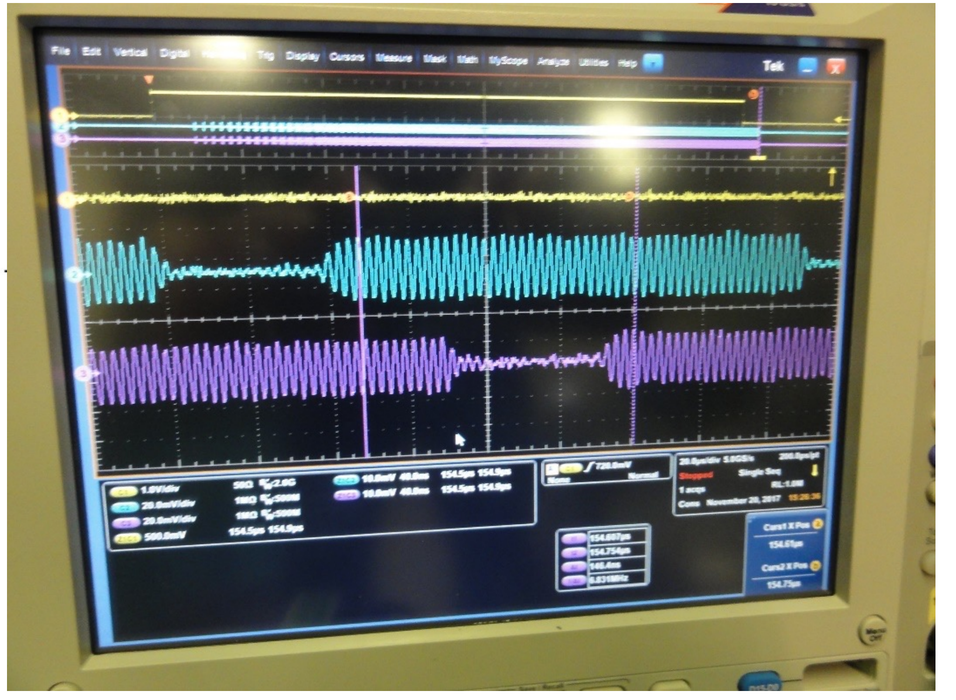
$$\beta = \frac{L}{\lambda\left(N + \frac{\varphi_{loop2} - \varphi_{loop1} + \Delta\varphi_{corr}}{2\pi}\right)}$$



Beam RF phases measured at delta-t loops.

Time-Of-Flight Measurement of Absolute Beam Energy





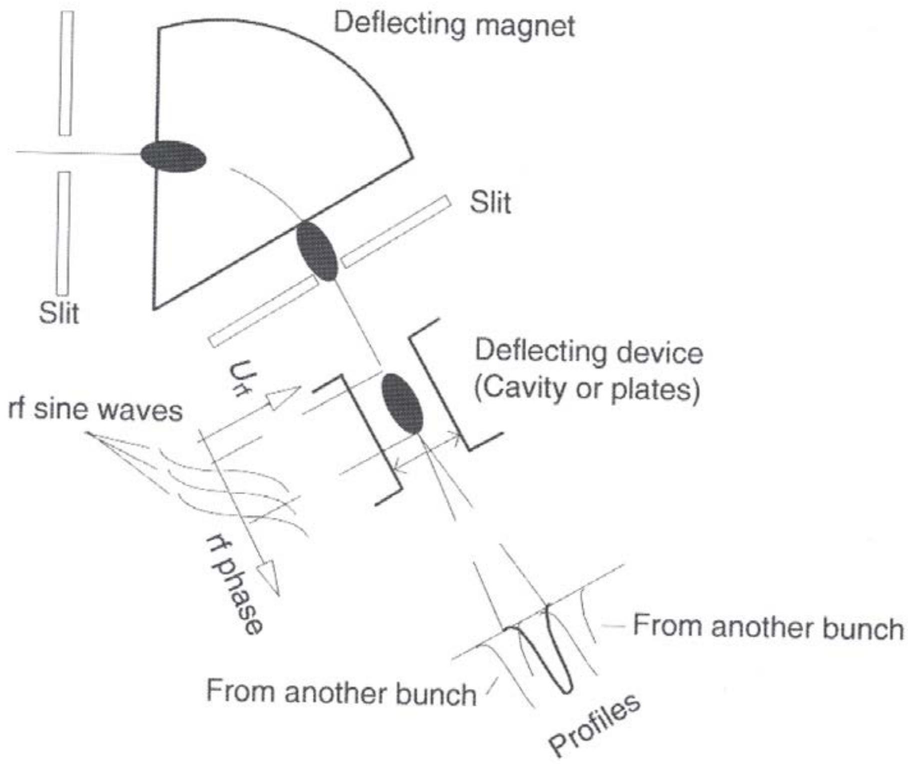
Beam velocity

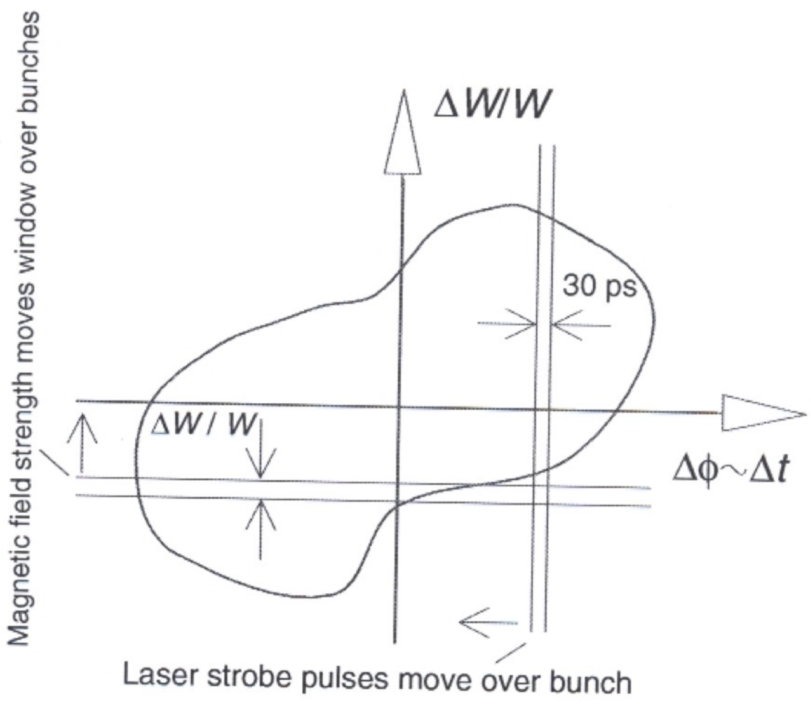
$$\beta = \frac{L}{c[t-(\tau_{cable2}-\tau_{cable1})]}$$

Beam energy

$$W = mc^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)$$

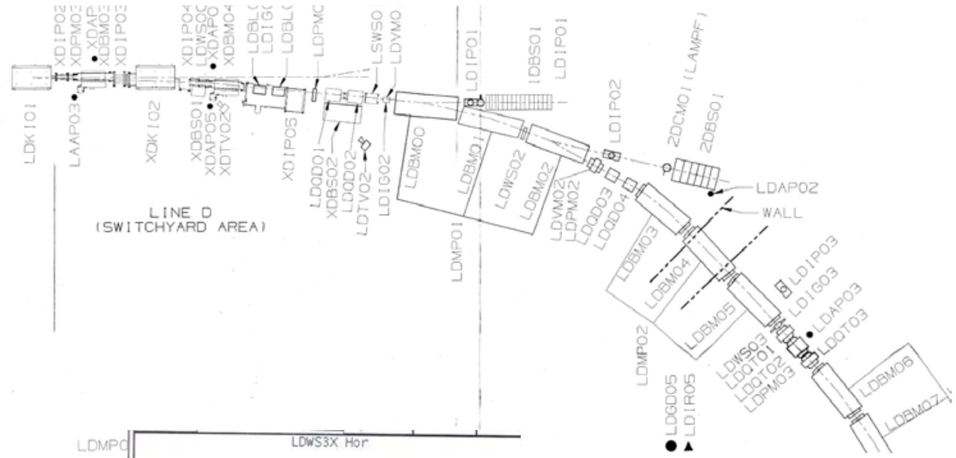
Longitudinal Beam Emittance Measurement



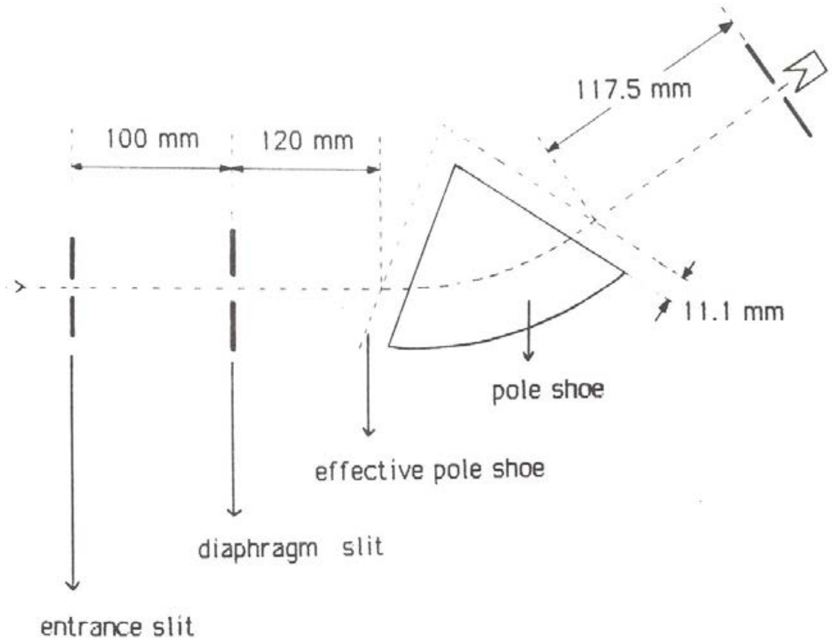


Measurement of Beam Energy Spread

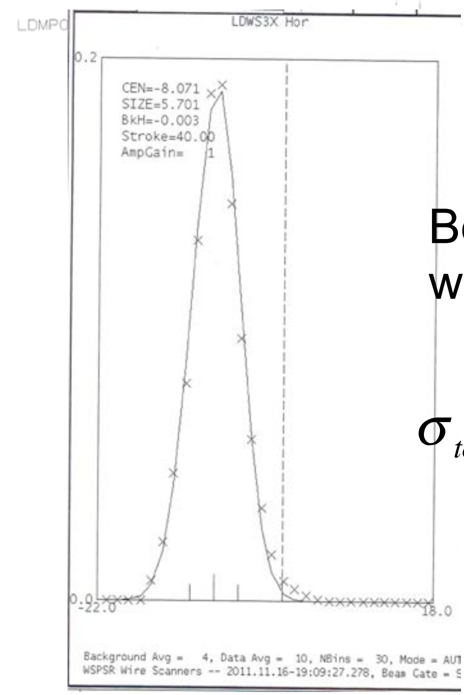
High-dispersive part of 800 MeV beamline



Faraday cup



Magnetic energy analyzer



Beam size in point with high dispersion:

$$\sigma_{tot} = \sqrt{\sigma + (\eta_{disp} \frac{\sigma_p}{p})^2}$$

Beam energy- spread-dependent wire scan

Bunch Shape Monitor

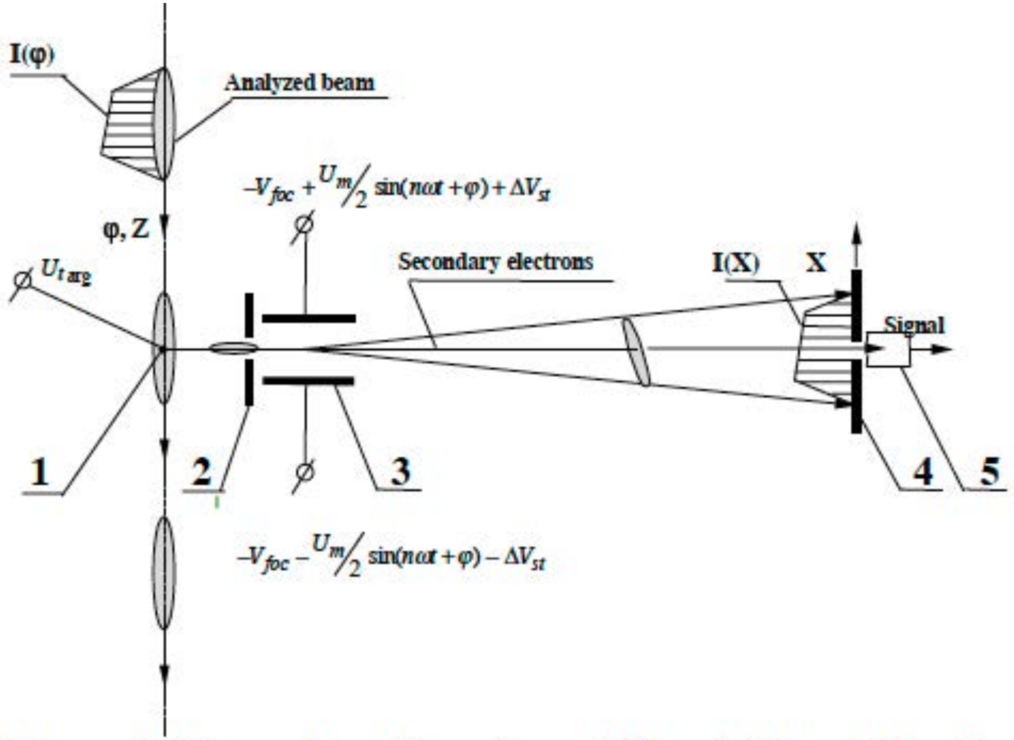


Figure 1: General configuration of Bunch Shape Monitor (1 –wire target, 2-input collimator, 3-deflector, 4-output collimator, 5-electron collector).

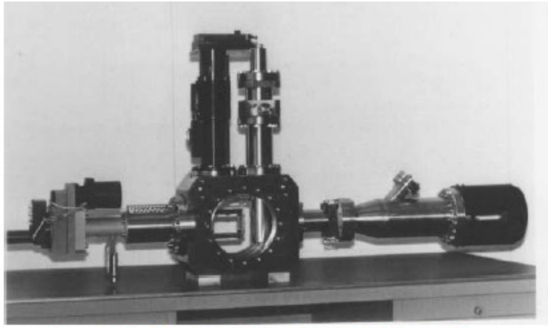


Figure 4: 3D-BSM for CERN Linac-2.

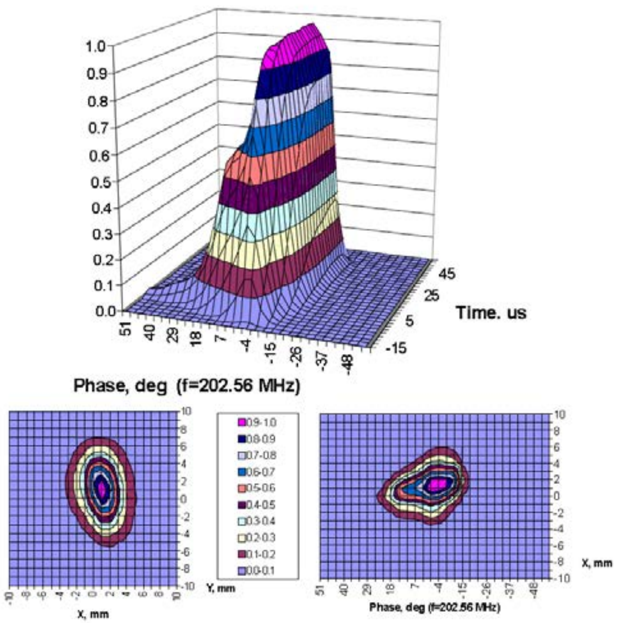


Figure 14: Behaviour of bunch shape in time, beam cross-section and longitudinally-transversal distribution measured at the exit of CERN Linac-2 with the 3D-BSM.

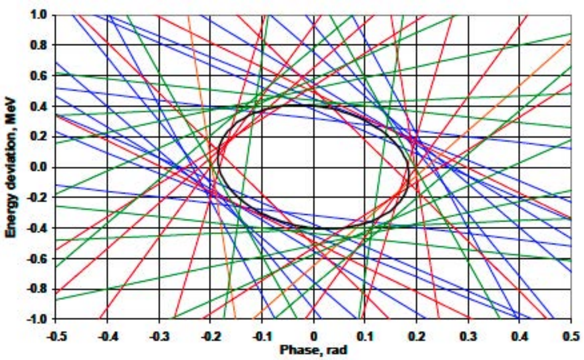


Figure 7: Bunch boundaries transformed to the entrance of CCL#1 and an equivalent phase ellipse.