Accelerator Physics
Statistical and Collective Effects II

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Lecture 14

Beam Temperature

- K-V has single value for the transverse Hamiltonian

$$\frac{1}{\varepsilon_x}\left(\frac{x^2+(\alpha_x x+\beta_x x')^2}{\beta_x}\right)+\frac{1}{\varepsilon_y}\left(\frac{y^2+(\alpha_y y+\beta_y y')^2}{\beta_y}\right)=C$$

$$\psi(x,x',y,y')\propto\delta(C-1)$$

No Temperature!

- Temperature in beam introduces thermal spreads
  - Transverse Temperature
    - Debye length
  - Longitudinal Temperature
    - Landau Damping

Warm Beam Models

- Model beam as fluid with pressure scalar
- Bennet model for self-pinched beam in plasma
  - Newton's law for fluid mass in volume *dV* (no electric field in plasma)

$$mn\frac{d\vec{v}}{dt}dV = \vec{F} = -\nabla p dV + en\vec{v}\times\vec{B}dV$$

  - In equilibrium

$$\nabla p = en\vec{v}\times\vec{B}dV$$

  - Equation for magnetic field

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \mu_0 e v_z n\left(r\right) \rightarrow B_\theta = \frac{\mu_0 e}{r}v_z\int_0^r n\left(r'\right)r'dr'$$

Bennet Profile

- Using the ideal gas law the density solves

$$\frac{dn}{dr}kT = -n\frac{\mu_0 e}{r}v_z^2\int_0^r n(r')r'dr'$$

- Bennet profile

$$n(r) = \frac{n_0}{\left(1 + r^2 / r_b^2\right)^2}$$

- Beam size

$$r_b^2 = \frac{8kT}{\beta_z^2 n_0 e^2 / \varepsilon_0}$$

Beam Profile Long Solenoid

– Radial focusing in rotating frame

$$\ddot{r} = -nm\omega_L^2 r$$

– Using the ideal gas law the density solves

$$\frac{dn}{dr} kT = -nm\omega_L^2 r$$

– Gaussian profile

$$n(r) = n_0 \exp\left(-r^2 / 2\sigma_r^2\right)$$

– Beam size

$$\sigma_r^2 = \frac{kT}{m\omega_L^2}$$

Waterbag Distribution

- Lemons and Thode were first to point out SC field is solved as Bessel Functions for the 2D adiabatic equation of state (P/n^2 constant). Later, others, including my advisor and I showed the equation of state was exact for the waterbag transverse distribution.

$$H_T = \frac{p_z^2}{2m}\left(x'^2 + y'^2\right) + \frac{m\omega_0^2\left(x^2+y^2\right)}{2} + e\phi_{SC}$$

$$\psi = A\Theta\left(H_0 - H_T\right)$$

$$n\left(r\right) = \iint \psi \, dx'dy' = \hat{n}_b\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right) \qquad \phi_0 = H_0 / e$$

$$\sigma_v^2 = \frac{\iint \psi\, p_z^2\left(x'^2+y'^2\right) dx'dy'}{m^2 \iint \psi\, dx'dy'} = \frac{H_0}{m}\left(1 - \frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - \frac{\phi_{SC}}{\phi_0}\right)$$

Self-consistent potential solves

$$\nabla^2 \phi_{SC} - \frac{\phi_{SC}}{\lambda_D^2} = \frac{e\hat{n}_b}{\varepsilon_0}\left[\frac{m\omega_0^2\left(x^2+y^2\right)}{2H_0} - 1\right]$$

$$\lambda_D = \frac{\sigma_v}{\omega_p} = \sqrt{\frac{\varepsilon_0 m}{e^2\hat{n}_b}\frac{H_0}{m}} = \sqrt{\frac{\varepsilon_0 H_0}{e^2\hat{n}_b}}$$ Debye Length

Analytic solutions in terms of Modified Bessel Functions

$$e\phi_{SC}(r) = -\frac{m\omega_0^2\left(x^2+y^2\right)}{2} + A\left(I_0\left(r/\lambda_D\right)-1\right) + BK_0\left(r/\lambda_D\right)$$

$B = 0$ by boundary condition

A chosen so that solution without I_0 solution to inhomogeneous eqn.

Equation for Beam Radius

Now

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right]\frac{r^2}{2}=2$$

$$\therefore A = m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)$$

At $r = r_b$ the density vanishes

$$H_0 = m\lambda_D^2\left(2\omega_0^2-\omega_p^2\right)\left(1-I_0\left(r_b/\lambda_D\right)\right)$$

$$1+\frac{\omega_p^2}{2\omega_0^2-\omega_p^2}=I_0\left(r_b/\lambda_D\right)$$

$$n\left(r\right)=\hat{n}_b\frac{I_0\left(r_b/\lambda_D\right)-I_0\left(r/\lambda_D\right)}{I_0\left(r_b/\lambda_D\right)-1}$$

Debye Length Picture*

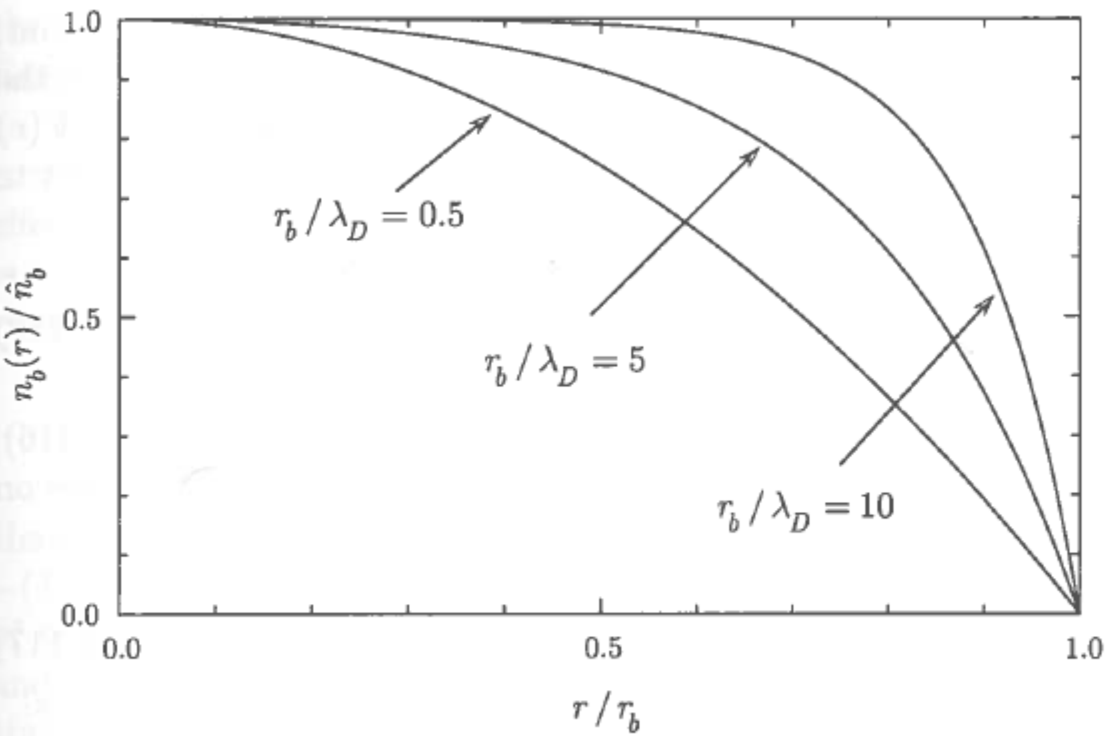


Figure 5.4. Plot of the normalized density profile $n_b(r)/\hat{n}_b$ versus r/r_b obtained from Eq. (5.115) for the choice of equilibrium distribution in Eq. (5.109). Here, the three cases correspond to the choices $r_b/\lambda_D = 0.5$, $r_b/\lambda_D = 5$ and $r_b/\lambda_D = 10$ [see Eq. (5.114)].

*Davidson and Qin

Collisionless (Landau) Damping

- Other important effect of thermal spreads in accelerator physics
- Longitudinal Plasma Oscillations (1 D)

$$\frac{\partial n}{\partial t} + \nabla \cdot v_z n = 0$$

$$\frac{dv_z}{dt} = \frac{-e}{m} E_z$$

$$\frac{\partial E_z}{\partial z} = \frac{-en}{\varepsilon_0}$$

Linearized

$$\frac{\partial \delta n}{\partial t} + n_0 \frac{\partial \delta v_z}{\partial z} = 0$$

$$\frac{\partial \delta v_z}{\partial t} = \frac{-e}{m} \delta E_z$$

$$\frac{\partial \delta E_z}{\partial z} = \frac{-e\delta n}{\varepsilon_0}$$

$$\frac{\partial^2 \delta n}{\partial t^2} = \frac{e n_0}{m} \frac{\partial \delta E_z}{\partial z} = -\frac{e^2 n_0}{\varepsilon_0 m} \delta n$$

$$\delta n \propto e^{\pm i\omega_p t} \qquad \omega_p = \sqrt{\frac{e^2 n_0}{\varepsilon_0 m}}$$

In fluid limit *plasma* oscillations are undamped

Vlasov Analysis of Problem

$$F_e(z, p_z, t) = \iint \Psi(z, \vec{p})\, dp_x dp_y$$

$$\left\{ \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + e \frac{\partial \phi}{\partial z} \frac{\partial}{\partial p_z} \right\} F_e = 0$$

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{e}{\varepsilon_0} \left(\int F_e dp_z - n_i \right)$$

0th order solution

$$F_e = F_0(p_z), \qquad \phi_0 = 0$$

linearized

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} \right) \delta F_e = -e \frac{\partial F_0}{\partial p_z} \frac{\partial \delta \phi}{\partial z}$$

$$\frac{\partial^2 \delta \phi}{\partial z^2} = \frac{e}{\varepsilon_0} \int \delta F_e dp_z$$

Initial Value Problem

- Laplace in t and Fourier in z

$$\hat{F}(\omega) = \int_0^\infty dt e^{i\omega t} F(t) \qquad \text{Im}\,\omega \text{ large enough to converge}$$

$$F(t) = \frac{1}{2\pi} \int_C d\omega e^{-i\omega t} \hat{F}(\omega)$$

$$\int_0^\infty dt e^{i\omega t} \frac{d}{dt} F(t) = -i\omega \hat{F}(\omega) - F(t=0)$$

$$\delta\phi(z,t) = \sum_{l=-\infty}^{\infty} \delta\hat{\phi}(l,t) e^{2\pi i l z/L}$$

$$\delta F_e(l, p_z, \omega) = \frac{i\delta F_e(l, p_z, t=0)}{\omega - v_z(2\pi l/L)} + \frac{e 2\pi l \partial F_0/\partial p_z}{L(\omega - v_z(2\pi l/L))} \delta\hat{\phi}(l,\omega)$$

$$\delta\hat{\phi}(l,\omega) = -\left(\frac{L}{2\pi l}\right)^2 \left[\frac{e^2}{\varepsilon_0} \int_{-\infty}^{\infty} dp_z \frac{2\pi l \partial F_0(p_z)/\partial p_z}{L(\omega - v_z(2\pi l/L))} \delta\hat{\phi}(l,\omega) + \frac{ei}{\varepsilon_0} \int_{-\infty}^{\infty} dp_z \frac{\delta F_e(l, p_z, t=0)}{\omega - v_z(2\pi l/L)} \right]$$

Dielectric function

- Landau (self-consistent) dielectric function

$$D(l,\omega)\,\delta\hat{\phi}(l,\omega) = N(l,\omega)$$

$$D(l,\omega) = 1 + \frac{e^2}{\varepsilon_0}\left(\frac{L}{2\pi l}\right)\int_{-\infty}^{\infty} dp_z \frac{\partial F_0(p_z)/\partial p_z}{\omega - v_z(2\pi l/L)}$$

- Solution for normal modes are

$$D(l,\omega) = 0$$

$$D(l,\omega) = 1 - \frac{e^2}{\varepsilon_0 m}\int_{-\infty}^{\infty} dp_z \frac{F_0(p_z)}{\left(\omega - v_z(2\pi l/L)\right)^2}$$

$$= 1 - \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{F_0(p_z)/n_0}{\left(\omega - v_z(2\pi l/L)\right)^2}$$

Collisionless Damping

- For Lorentzian distribution

$$\frac{F_0}{n_0} = \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

$$1 = \omega_p^2 \int_{-\infty}^{\infty} dp_z \frac{1}{\left(\omega - p_z\left(2\pi l / Lm\right)\right)^2} \frac{\Delta}{\pi\left[p_z^2 + \Delta^2\right]}$$

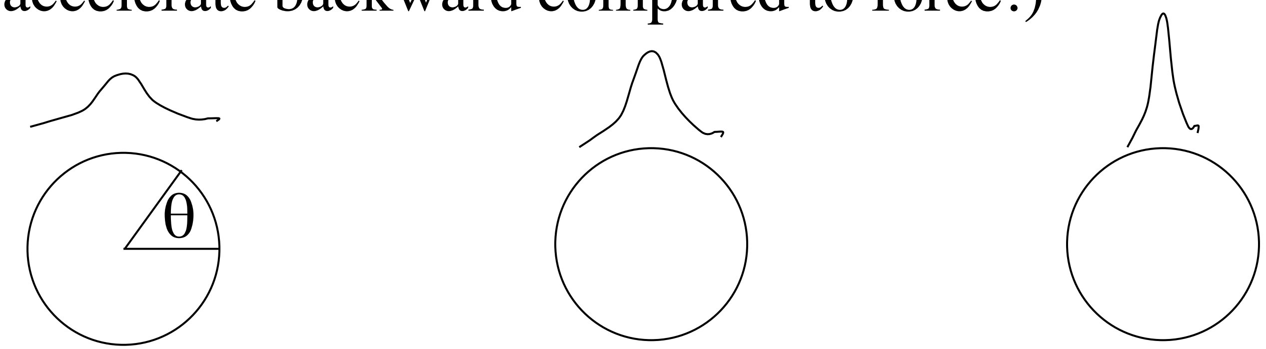
$$= \frac{\omega_p^2}{\left(\omega + i\left(2\pi l / Lm\right)\Delta\right)^2}$$

- Landau damping rate

$$\omega = \pm\omega_p - i\frac{2\pi l}{Lm}\Delta$$

Negative Mass Instability

- Simplified argument: assume longitudinal clump on otherwise uniform beam
- Particles pushed away from clump centroid
- If above transition, come back LATER if ahead of clump center and EARLIER if behind it
- The clump is therefore enhanced!
- INSTABILITY; particles act as if they have negative mass (they accelerate backward compared to force!)



Longitudinal Impedance

$W_{||}$ longitudinal wake function

ξ distance between exciting charge q and test charge

$$W_{||}(\xi) \equiv \frac{1}{q} \int_{ring} E_z\left(z, t_{q\ arrival} + \xi / \beta c\right) dz$$ units V/C

trailing particle (singly charged) picks up voltage per turn of

$$\Delta V(\bar{z}) = -e \int_{\bar{z}}^{"\infty"} \lambda(z) W_{||}(z - \bar{z})\, dz$$

total energy loss

$$\Delta U = -\int_{-\infty}^{\infty} e\lambda(\bar{z})\, d\bar{z} \int_{\bar{z}}^{"\infty"} e\lambda(z) W_{||}(z - \bar{z})\, dz$$

Frequency Domain

$I(\bar{z},t) = \beta c \lambda(\bar{z},t)$ note the coordinate $\bar{z}$ moves with beam

$$\Delta V(\bar{z},t) = -\frac{1}{\beta c} \int_{\bar{z}}^{"\infty"} I\left(z, t + \frac{z-\bar{z}}{\beta c}\right) W_{\|}(z-\bar{z})\, dz$$

Fourier Transform

$$\Delta V(\omega) = -I(\omega) \frac{1}{\beta c} \int_{\bar{z}}^{"\infty"} e^{-i\omega\xi/\beta c} W_{\|}(\xi)\, dz \equiv -Z_{\|}(\omega) I(\omega)$$

$$Z_{\|}(\omega) = \frac{1}{\beta c} \int_{-\infty}^{\infty} e^{-i\omega\xi/\beta c} W_{\|}(\xi)\, d\xi$$

$$W_{\|}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega z/\beta c} Z_{\|}(\omega)\, d\omega$$

Loss factor

$$k = \frac{\Delta U}{q^2} = \frac{2}{q^2} \int_0^{\infty} \mathrm{Re}\left[Z(\omega)\right] |I|^2(\omega)\, d\omega$$

NMI Simple Analysis

ω revolution frequency of particle

$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial t} + \frac{\partial\omega}{\partial\theta}\frac{\partial\theta}{\partial t}$$

$$\frac{d\omega}{dt} = \frac{d\omega}{dE}\frac{dE}{dt} = \frac{\eta_c\omega_0}{\beta^2 E_0}\frac{dE}{dt}$$

$$\frac{dE}{dt} = qV_{zn}\frac{\omega_0}{2\pi} = -qZ_{\|}I_n e^{i(n\theta-\Omega t)}\frac{\omega_0}{2\pi}$$

$\omega = \omega_0 + \omega_n e^{i(n\theta-\Omega t)}$ Ω oscillation frequency of disturbance

$$\omega_n\left(\Omega - n\omega_0\right) = -i\frac{q\eta_c\omega_0^2}{2\pi\beta^2}\frac{Z_{\|}I_n}{E_0}$$

Linearized Continuity Equation

$$I = v_z \rho \pi r_b^2 = v_z \lambda$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}\left(v_z \rho\right) = 0$$

$$\frac{\partial \lambda}{\partial t} + \frac{1}{R}\frac{\partial}{\partial \theta}\left(v_z \lambda\right) =$$

$$\frac{\partial \delta\lambda}{\partial t} + \omega_0 \frac{\partial \delta\lambda}{\partial \theta} + \lambda_0 \frac{\partial \delta\omega}{\partial \theta} = 0$$

$$\left(\Omega - n\omega_0\right) I_n = \omega_n n I_0$$

Oscillation Frequency

$$\Delta\Omega^2 = (\Omega - n\omega_0)^2 = -i\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0} Z_{||}$$

$\mathrm{Re}\, Z_{||} \neq 0 \rightarrow 1$ mode has positive imaginary part

$\rightarrow$ instability

Resistive impedance has positive real part

"Resistive wall instability"

If $\mathrm{Re}\, Z_{||} = 0$ (e.g. space charge impedance at long wavelengths)

stability/instability depends on sign of RHS

$\mathrm{Im}\, Z_{||} < 0$ (inductive, stable if $\eta_c < 0$, unstable if $\eta_c > 0$)

$\mathrm{Im}\, Z_{||} > 0$ (capacitive, space charge is this way,

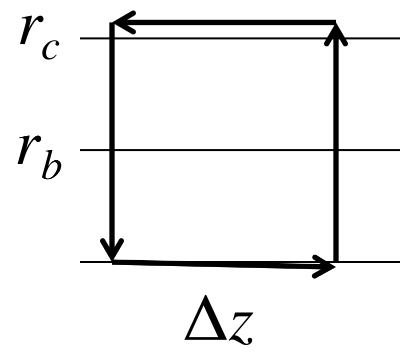
stable if $\eta_c > 0$, unstable if $\eta_c < 0$)

Later case is negative mass instability

NMI Growth time

Impedance?

$$E_r = \begin{cases} \dfrac{e\lambda r}{2\pi\varepsilon_0 r_b^2} & r < r_b \\ \dfrac{e\lambda}{2\pi\varepsilon_0 r} & r > r_b \end{cases} \qquad B_\theta = \begin{cases} \beta c \dfrac{\mu_0 e\lambda r}{2\pi r_b^2} & r < r_b \\ \beta c \dfrac{\mu_0 e\lambda}{2\pi r} & r > r_b \end{cases}$$



$$E_z = -\frac{e}{4\pi\varepsilon_0}\left(1-\beta^2\right)\frac{\partial\lambda}{\partial z}\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\lambda \propto \lambda_n e^{i(n\theta-\Omega t)}$$

$$V_{SC} = \frac{-in}{2\varepsilon_0\gamma^2}\lambda_n\left(1+2\ln\left(r_c/r_b\right)\right) = \frac{-in}{2\varepsilon_0\gamma^2\beta c}I_n\left(1+2\ln\left(r_c/r_b\right)\right)$$

$$\left(\Omega - n\omega_0\right)^2 = -i\frac{nq\eta_c\omega_0^2 I_0}{2\pi\beta^2 E_0}Z_{SC} = -\frac{n^2\omega_0^2}{\beta^2\gamma^2}\left(\frac{q\eta_c I_0}{4\pi\varepsilon_0 c\beta E_0}\left(1+2\ln\left(r_c/r_b\right)\right)\right)$$

Stabilization by Beam Temperature?

Canonical variables $\theta, \delta \equiv \Delta p / p_0$

$$\left[\frac{\partial}{\partial t} + \dot{\theta}\frac{\partial}{\partial \theta} + \dot{\delta}\frac{\partial}{\partial \delta}\right]\psi = 0$$

$$\psi = \psi_0 + \psi_n e^{i(n\theta - \omega_n t)}$$

$$i(\omega_n - n\omega)\psi_n = \frac{\dot{\delta}}{e^{i(n\theta - \omega_n t)}}\frac{\partial \psi_0}{\partial \delta}$$

$$\frac{\partial \psi_0}{\partial \delta} = \frac{\partial \psi_0}{\partial \omega}\frac{\partial \omega}{\partial \delta} = \eta_c \omega_0 \frac{\partial \psi_0}{\partial \omega}$$

current perturbation is

$$I_n = q\omega_0 \int_{-\infty}^{\infty} \psi_n d\delta$$

Dispersion Relation

$$\psi_0(\delta) = \eta_c \omega_0 \Phi_0(\omega)$$

$$\dot{\delta} = \frac{1}{\eta_c \omega_0} \dot{\omega} = \left(\frac{dE}{dt} \right) / \left(\beta^2 E_0 \right)$$

$$1 = i \frac{q^2 \omega_0^3 \eta_c Z_{\|}}{2\pi \beta^2 E_0} \int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega$$

recover before

$$\Phi_0 = N_b \delta(\omega - \omega_0) / 2\pi$$

$$\int \frac{\partial \Phi_0 / \partial \omega}{\omega_n - n\omega} d\omega = -\frac{N_b n}{2\pi \left(\omega_n - n\omega_0 \right)^2}$$

Landau Damping

Use our favorite analytic distribution

$$\psi_0(\delta) \propto \frac{1}{\pi} \frac{\delta_0}{\delta_0^2 + \delta^2} \qquad \Phi_0(\omega) \propto \frac{1}{\pi} \frac{\hat{\omega}}{\hat{\omega}^2 + (\omega - \omega_0)^2}$$

$$\hat{\omega} = \delta_0 \eta_c \omega_0$$

$$1 = -i \frac{q \omega_0^2 \eta_c Z_{\|} I_0 n}{2\pi\beta^2 E_0} \int \frac{\hat{\omega}}{(\omega_n - n\omega)^2 \pi \left(\hat{\omega}^2 + (\omega - \omega_0)^2\right)} d\omega$$

$$1 = -i \frac{q \eta_c Z_{\|} I_0 n}{2\pi\beta^2 E_0} \frac{\omega_0^2}{(\omega_n - n\omega_0 + ni\hat{\omega})^2}$$

$$\omega_n = n\omega_0 - ni\hat{\omega} + \sqrt{V + iU}$$

LD from another view

Single Oscillator

$$\ddot{u} + \Omega^2 u = F e^{-i\omega t}$$

$$u(t) = \frac{F e^{-i\omega t}}{2\omega}\left(\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right)$$

Many oscillators distributed in frequency simultaneously excited

$$\psi(\Omega) = \frac{1}{N}\frac{dN}{d\Omega}$$

$$U = \frac{\sum_{i=1}^{N} u_i}{N}$$

$$U = \frac{F e^{-i\omega t}}{2\omega}\int\left[\frac{1}{\Omega - \omega} - \frac{1}{\Omega + \omega}\right] d\Omega$$

for $\psi(\Omega) = \psi(-\Omega)$

$$U = \frac{F e^{-i\omega t}}{\omega}\int \frac{\psi(\Omega)}{\Omega - \omega} d\Omega$$

Resonance Effect

$$U=\frac{Fe^{-i\omega t}}{\omega}\left[+i\pi\psi(\omega)+P.V.\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega\right]$$

$$\dot{U}=Fe^{-i\omega t}\left[\pi\psi(\omega)-iP.V.\int\frac{\psi(\Omega)}{\Omega-\omega}d\Omega\right]$$

For our analytic Lorentzian

$$\psi(\Omega)=\frac{\Delta}{\pi\left(\Delta^2+\Omega^2\right)}$$

$$\dot{U}=\frac{Fe^{-i\omega t}}{-i\Delta-\omega}=\frac{Fe^{-i\omega t}}{\Delta^2+\omega^2}(\Delta-i\omega)$$

Energy goes in!

Where does it go?

Inhomogeneous Solution

$$u(t) = a \sin \Omega t + \frac{F}{\Omega^2 - \omega^2} \sin \omega t$$

Solution with zero initial excitation

$$a = -\frac{\omega}{\Omega} \frac{F}{\Omega^2 - \omega^2}$$

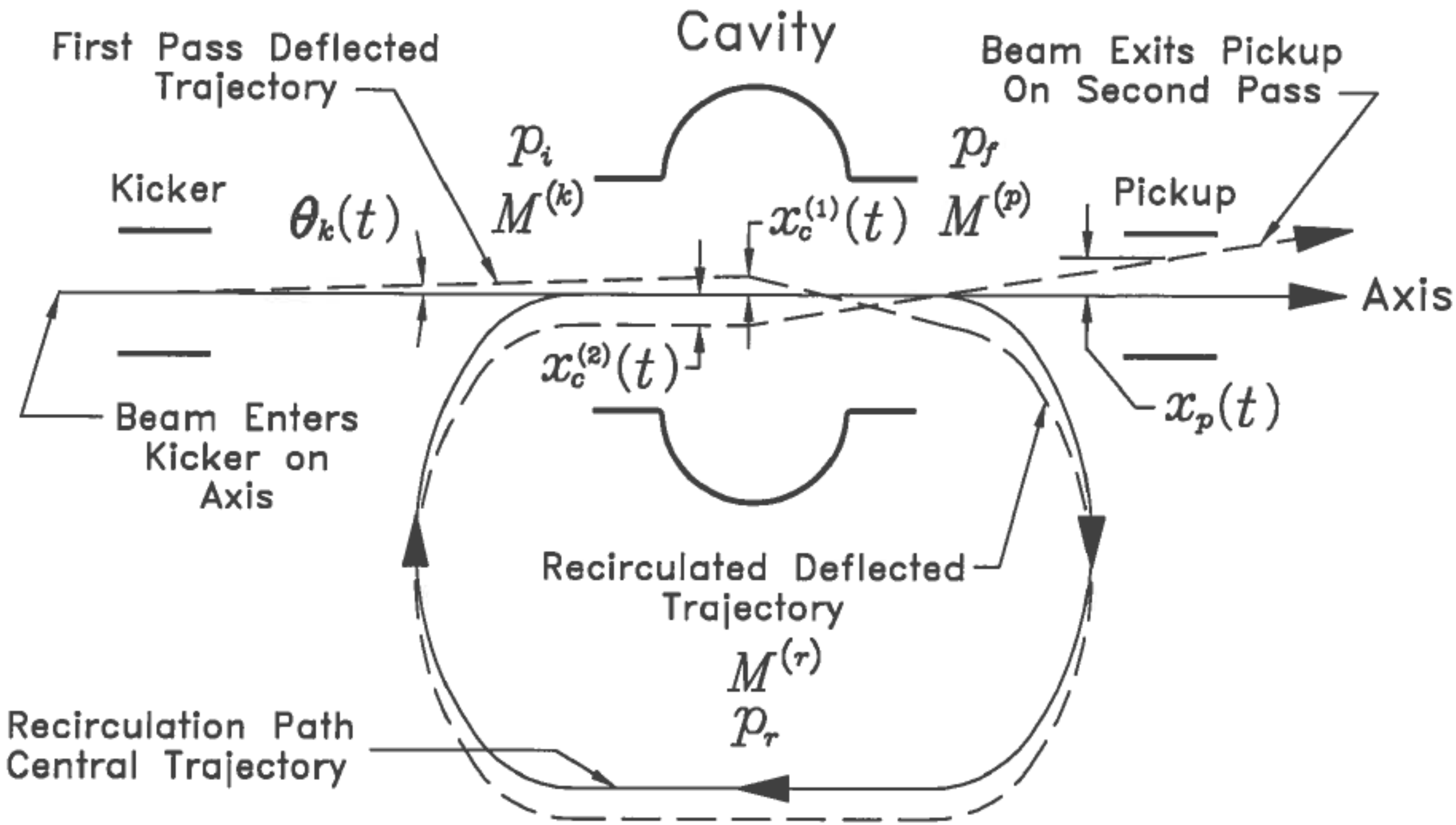
$$\therefore u_{\Omega \neq \omega} = \frac{F}{\Omega^2 - \omega^2} \left(\sin \omega t - \frac{\omega}{\Omega} \sin \Omega t \right)$$

No energy flow

$$\therefore u_{\Omega = \omega} = \frac{F}{\Omega^2 - \omega^2} \left(t \cos \omega t - \frac{\sin \omega t}{\omega} \right)$$

Resonant particles capture energy and oscillation generated out of phase

Multipass BBU Instability



BBU Theory

following Krafft, Laubach, and Bisognano

$$W_{transverse}(\tau) = \left(\frac{R}{Q}\right)_{HOM} \frac{k_{HOM}\omega_{HOM}}{2} e^{-\omega_{HOM}\tau/2Q_{HOM}} \sin \omega_{HOM}\tau$$

Single cavity/Single HOM case

$$V(t) = \int_{-\infty}^{t} W_{transverse}(t-t')I(t')d(t')dt'$$

On the second pass

$$d(t) = \frac{T_{12}eV(t-t_r)}{c}$$

With no initial displacement

$$V(t) = \frac{T_{12}e}{c} \int_{-\infty}^{t} W_{transverse}(t-t')I(t')V(t'-t_r)dt'$$

Delay differential (integral) equation

Beam current

$$I(t)=\sum_{m=-\infty}^{\infty} I_0 t_0 \delta(t-mt_0)$$

Normal mode

$$V(nt_0)=V_0 e^{-i\omega n t_0}$$

Sum the geometric series for eigenvalue equation

$$1=Ke^{i\omega t_r}\frac{e^{i\omega t_0}e^{-\omega_{HOM}t_0/2Q_{HOM}}\sin\omega_{HOM}t_0}{1+\left(e^{i\omega t_0}e^{-\omega_{HOM}t_0/2Q_{HOM}}\right)^2-2e^{i\omega t_0}e^{-\omega_{HOM}t_0/2Q_{HOM}}\cos\omega_{HOM}t_0}$$

$$K=(R/Q)_{HOM}\,k_{HOM}^2 e T_{12} I_0 t_0/2$$

For $T_{12}\sin\omega_{HOM}t_r<0\rightarrow K\ll 1$ at threshold

Perturbation Theory Works

$$e^{i\omega t_0} e^{-\omega_{HOM} t_0 / 2Q_{HOM}} \doteq \left[1 \mp \frac{iK}{2} e^{i\omega t_r} \right] e^{\pm i\omega_{HOM} t_0}$$

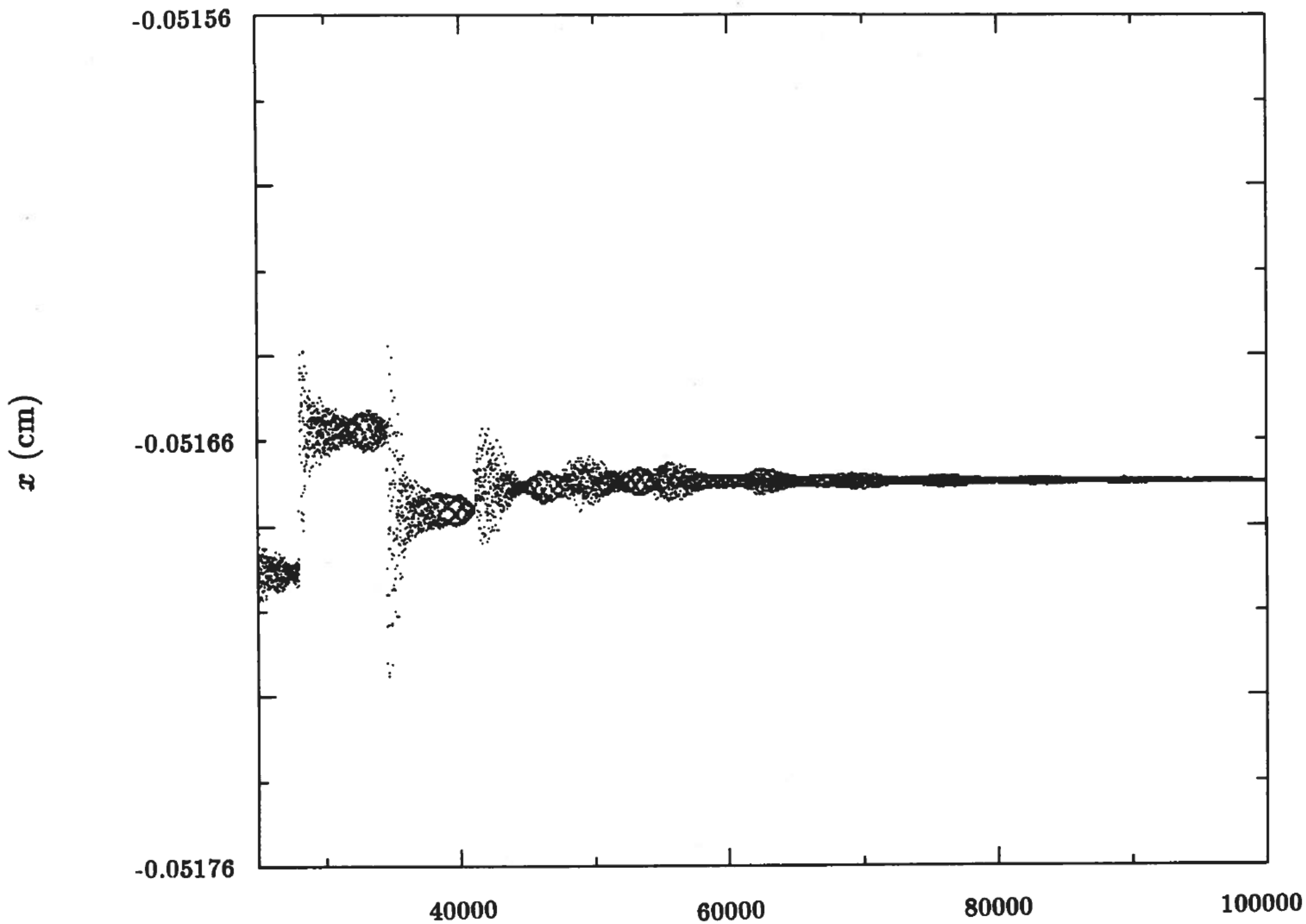
Growth rate

$$\mathrm{Im}(\omega) \doteq -\frac{K \sin(\omega_{HOM} t_r)}{2t_0} - \frac{\omega_{HOM}}{2Q_{HOM}}$$

Threshold current

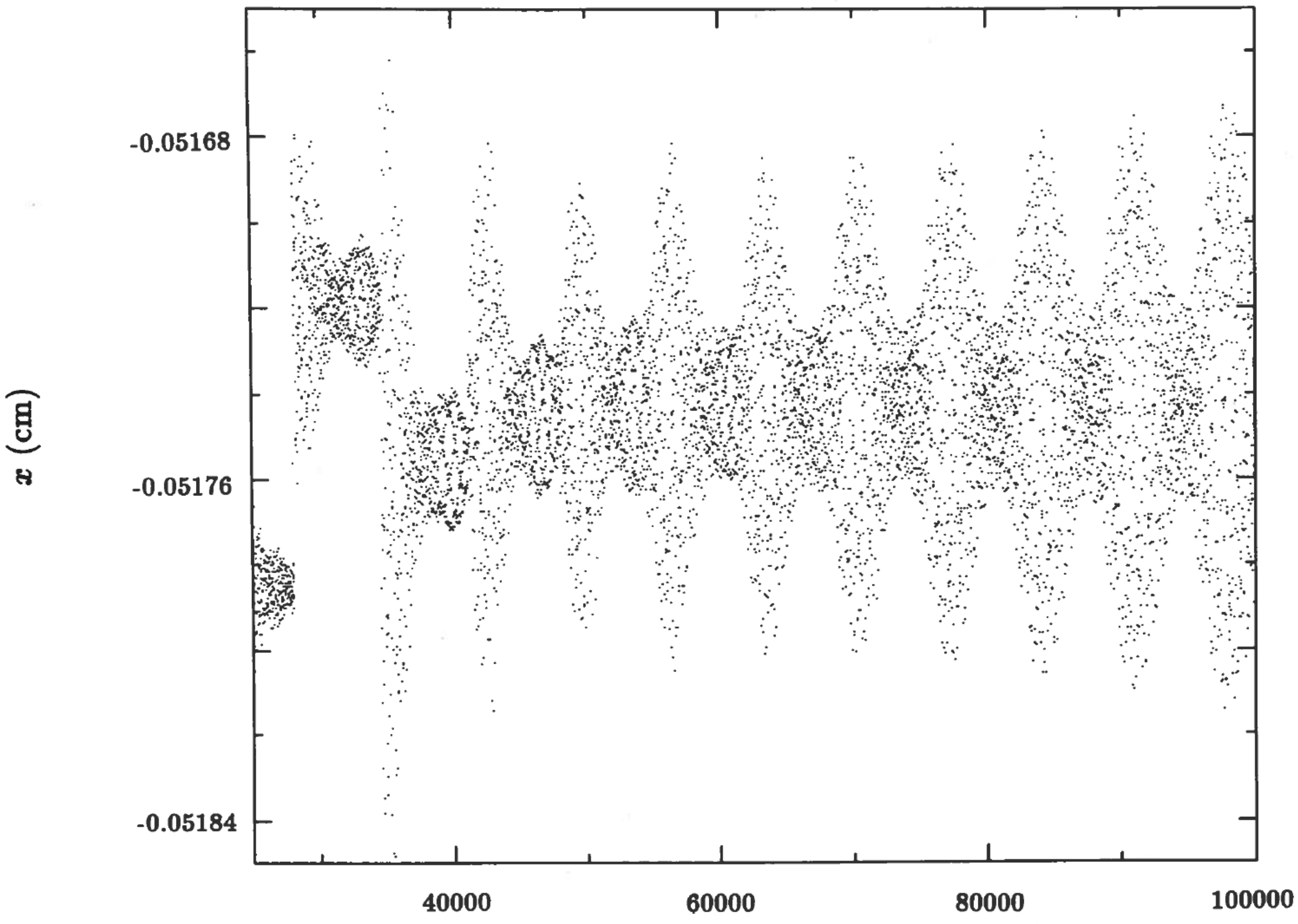
$$I_{th} = \frac{1}{e} \frac{2\omega_{HOM}}{(R/Q)_{HOM} Q_{HOM} k_{HOM}^2 \left| T_{12} \sin(\omega_{HOM} t_r) \right|}$$

CEBAF (Design) Simulations



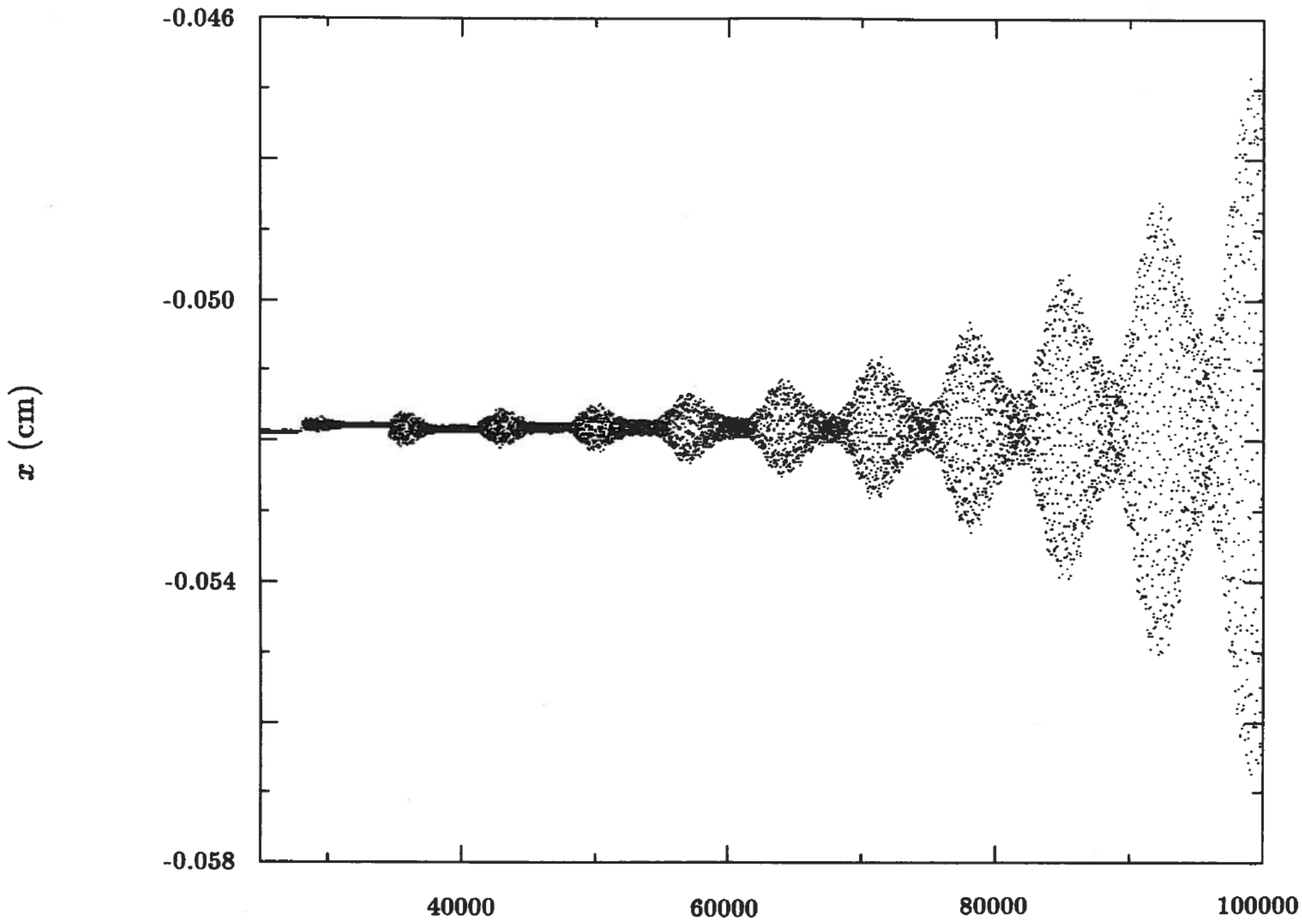
Bunch Number

Stable



Bunch Number

Close to threshold



Bunch Number

Unstable

Chromatic (Landau) Damping

When T_{12} depends on energy offset δ, threshold current is modified to

$$I_{th} = \frac{1}{e} \frac{2\omega_{HOM}}{(R/Q)_{HOM} Q_{HOM} k_{HOM}^2 \left| T_{12,eff} \sin(\omega_{HOM} t_r) \right|}$$

$$T_{12,eff} = \frac{\int T_{12}(\delta) f(\delta) d\delta}{\int f(\delta) d\delta}$$

If $\xi\delta \gg 1,\ T_{12,eff} \to 0$