Accelerator Physics
Linear Optics

S. A. Bogacz, G. A. Krafft, S. De Silva, I. Neththikumara
Jefferson Lab
Old Dominion University
TA: Cannon Coats (Texas A&M)

Lecture 4

Linear Beam Optics Outline

- Particle Motion in the Linear Approximation
- Some Geometry of Ellipses
- Ellipse Dimensions in the β-function Description
- Area Theorem for Linear Transformations
- Phase Advance for a Unimodular Matrix
  - Formula for Phase Advance
  - Matrix Twiss Representation
  - Invariant Ellipses Generated by a Unimodular Linear Transformation
- Detailed Solution of Hill's Equation
  - General Formula for Phase Advance
  - Transfer Matrix in Terms of β-function
  - Periodic Solutions
- Non-periodic Solutions
  - Formulas for β-function and Phase Advance
- Beam Matching

Linear Particle Motion

Fundamental Notion: The *Design Orbit* is a path in an Earth-fixed reference frame, i.e., a differentiable mapping from [0,1] to points within the frame. As we shall see as we go on, it generally consists of *arcs of circles* and *straight lines*.

$$\sigma : [0,1] \to \mathrm{R}^3$$

$$\sigma \quad \to \vec{X}(\sigma) = \left(X(\sigma), Y(\sigma), Z(\sigma) \right)$$

Fundamental Notion: *Path Length*

$$ds = \sqrt{\left(\frac{dX}{d\sigma}\right)^2 + \left(\frac{dY}{d\sigma}\right)^2 + \left(\frac{dZ}{d\sigma}\right)^2}\, d\sigma$$

The *Design Trajectory* is the path specified in terms of the path length in the Earth-fixed reference frame. For a relativistic accelerator where the particles move at the velocity of light, $L_{tot}=ct_{tot}$.

$$s:[0,L_{tot}]\rightarrow \mathrm{R}^3$$

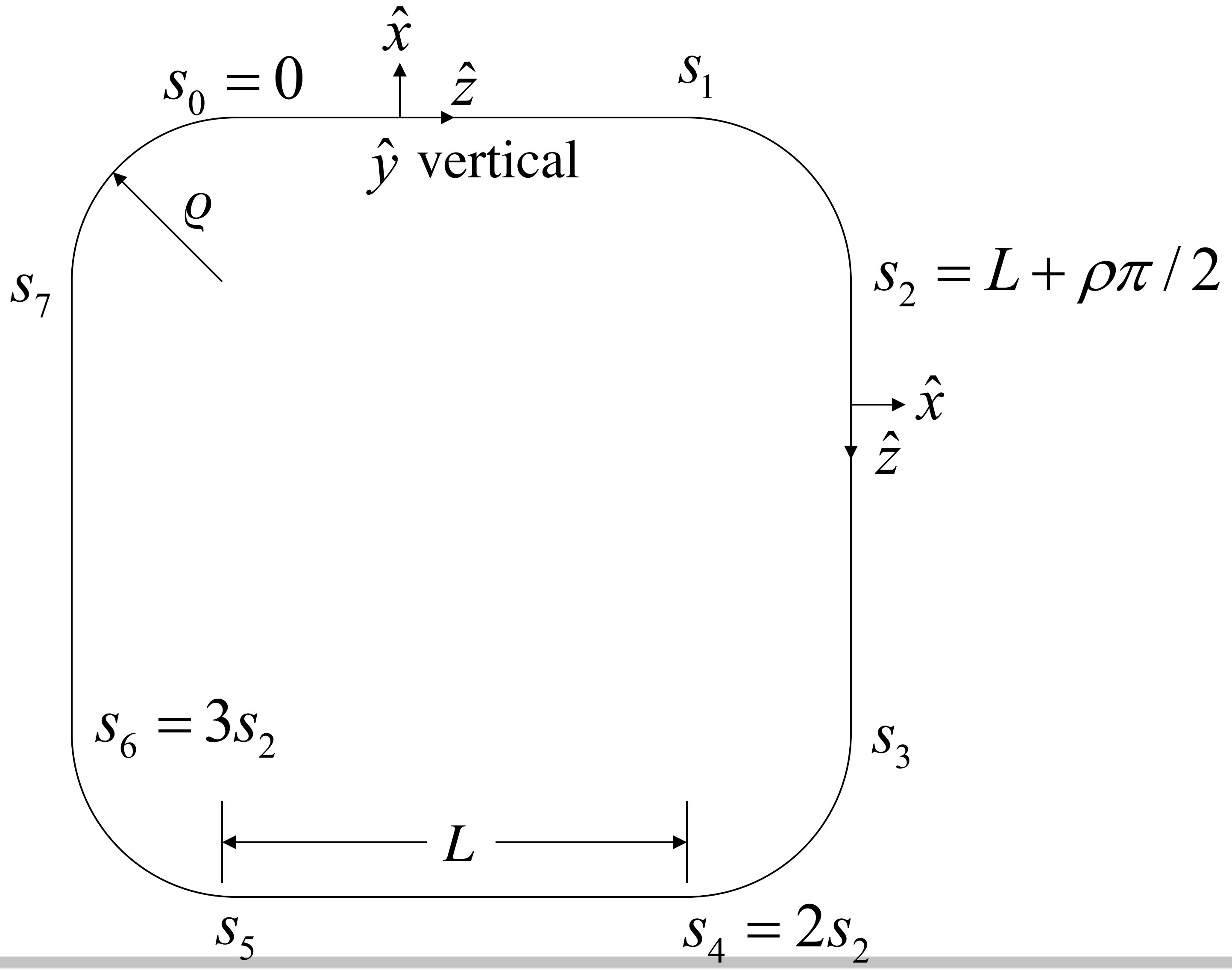
$$s \;\rightarrow\; \vec{X}(s)=\left(X(s),Y(s),Z(s)\right)$$

The first step in designing any accelerator, is to specify bending magnet locations that are consistent with the arc portions of the Design Trajectory.

Comment on Design Trajectory

The notion of specifying curves in terms of their path length is standard in courses on the vector analysis of curves. A good discussion in a Calculus book is Thomas, *Calculus and Analytic Geometry*, 4^{th} Edition, Articles 14.3-14.5. Most vector analysis books have a similar, and more advanced discussion under the subject of “Frenet-Serret Equations”. Because all of our design trajectories involve only arcs of circles and straight lines (dipole magnets and the drift regions between them define the orbit), we can concentrate on a simplified set of equations that “only” involve the radius of curvature of the design orbit. It may be worthwhile giving a simple example.

4-Fold Symmetric Synchrotron



Its Design Trajectory

$\left(0,0,s\right)$	$0<s<L=s_1$
$\left(0,0,L\right)+\rho\left(\cos\left(\left(s-s_1\right)/\rho\right)-1,0,\sin\left(\left(s-s_1\right)/\rho\right)\right)$	$s_1<s<s_2$
$\left(-\rho,0,L+\rho\right)+\left(s-s_2\right)\left(-1,0,0\right)$	$s_2<s<s_3$
$\left(-L-\rho,0,L+\rho\right)+\rho\left(-\sin\left(\left(s-s_3\right)/\rho\right),0,\cos\left(\left(s-s_3\right)/\rho\right)-1\right)$	$s_3<s<s_4$
$\left(-L-2\rho,0,L\right)+\left(s-s_4\right)\left(0,0,-1\right)$	$s_4<s<s_5$
$\left(-L-2\rho,0,0\right)+\rho\left(1-\cos\left(\left(s-s_5\right)/\rho\right),0,-\sin\left(\left(s-s_5\right)/\rho\right)\right)$	$s_5<s<s_6$
$\left(-L-\rho,0,-\rho\right)+\left(s-s_6\right)\left(1,0,0\right)$	$s_6<s<s_7$
$\left(-\rho,0,-\rho\right)+\rho\left(\sin\left(\left(s-s_7\right)/\rho\right),0,1-\cos\left(\left(s-s_7\right)/\rho\right)\right)$	$s_7<s<4s_2$

Betatron Design Trajectory

$$s:[0,2\pi R]\rightarrow \mathrm{R}^3$$

$$s \;\rightarrow\; \vec{X}(s)=\left(R\cos(s/R),R\sin(s/R),0\right)$$

Use path length *s* as independent variable instead of *t* in the dynamical equations.

$$\frac{d}{ds}=\frac{1}{\Omega_c R}\frac{d}{dt}$$

Betatron Motion in s

$$\frac{d^2 \delta r}{dt^2} + (1-n)\Omega_c^2 \delta r = \Omega_c^2 R \frac{\Delta p}{p}$$

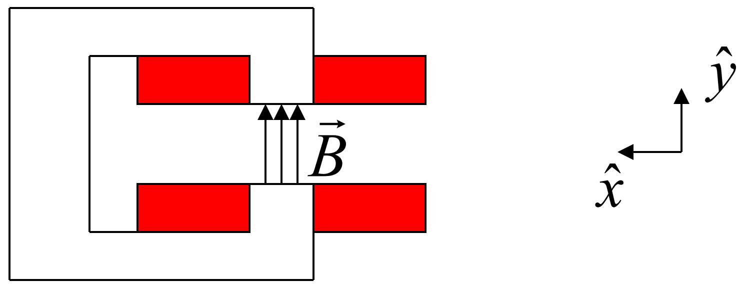
$$\frac{d^2 \delta z}{dt^2} + n\Omega_c^2 \delta z = 0$$

$$\Downarrow$$

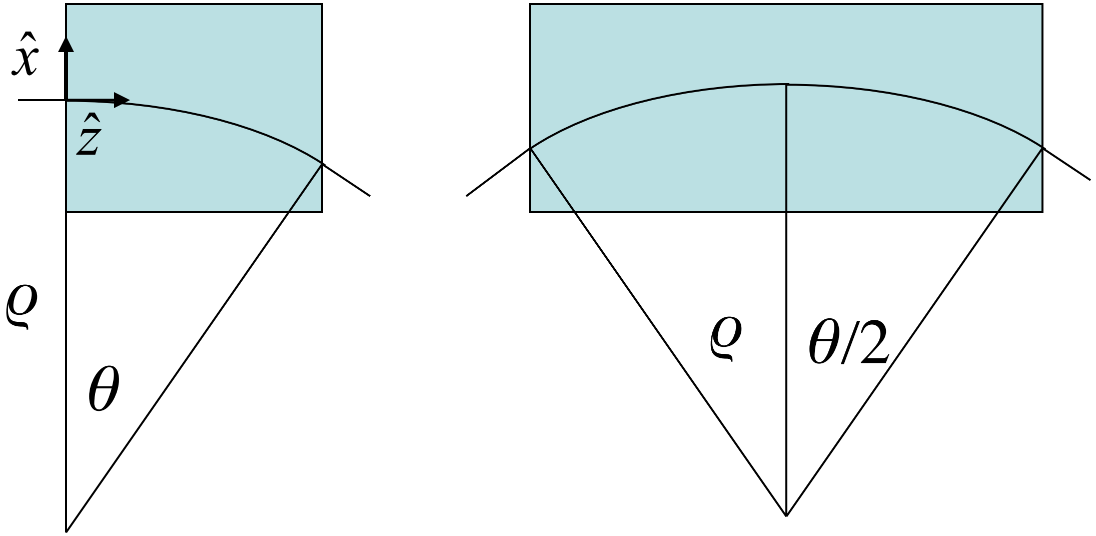
$$\frac{d^2 \delta r}{ds^2} + \frac{(1-n)}{R^2} \delta r = \frac{1}{R} \frac{\Delta p}{p}$$

$$\frac{d^2 \delta z}{ds^2} + \frac{n}{R^2} \delta z = 0$$

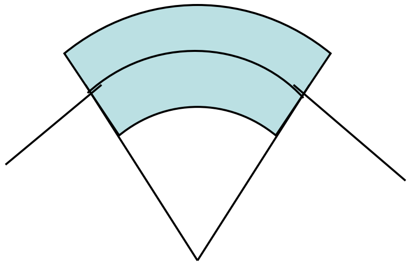
Bend Magnet Geometry



Rectangular Magnet of Length L



Sector Magnet



Bend Magnet Trajectory

For a uniform magnetic field

$$\frac{d(\gamma m\vec{V})}{dt} = \left[\vec{E} + \vec{V} \times \vec{B}\right]$$

$$\frac{d(\gamma m V_x)}{dt} = -qV_z B_y$$

$$\frac{d(\gamma m V_z)}{dt} = qV_x B_y$$

$$\frac{d^2 V_x}{dt^2} + \Omega_c^2 V_x = 0 \qquad \frac{d^2 V_z}{dt^2} + \Omega_c^2 V_z = 0$$

For the solution satisfying boundary conditions: $\vec{X}(0) = 0 \qquad \vec{V}(0) = V_{0z}\hat{z}$

$$X(t) = \frac{p}{qB_y}\left(\cos\left(\Omega_c t\right) - 1\right) = \rho\left(\cos\left(\Omega_c t\right) - 1\right) \qquad \Omega_c = qB_y / \gamma m$$

$$Z(t) = \frac{p}{qB_y}\sin\left(\Omega_c t\right) = \rho \sin\left(\Omega_c t\right)$$

Magnetic Rigidity

The magnetic rigidity is:

$$B\rho = \left|B_y \rho\right| = \frac{p}{\left|q\right|}$$

It depends only on the particle momentum and charge, and is a convenient way to characterize the magnetic field. Given magnetic rigidity and the required bend radius, the required bend field is a simple ratio. Note particles of momentum 100 MeV/c have a rigidity of 0.334 T m.

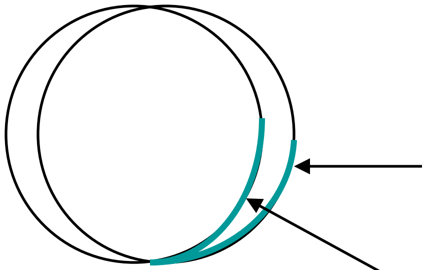
Long Dipole Magnet

$$BL = B\rho\left(2\sin\left(\theta/2\right)\right)$$

Normal Incidence (or exit)
Dipole Magnet

$$BL = B\rho\sin\left(\theta\right)$$

Natural Focusing in Bend Plane



Can show that for either a displacement perturbation or angular perturbation from the design trajectory

$$\frac{d^2x}{ds^2} = -\frac{x}{\rho_x^2(s)} \qquad \frac{d^2y}{ds^2} = -\frac{y}{\rho_y^2(s)}$$

Quadrupole Focusing

$$\vec{B}(x,y) = B'(s)(x\hat{y} + y\hat{x})$$

$$\gamma m \frac{d\mathrm{v}_{\mathrm{x}}}{ds} = -qB'(s)x \qquad \gamma m \frac{d\mathrm{v}_{\mathrm{y}}}{ds} = qB'(s)y$$

$$\frac{d^2x}{ds^2} + \frac{B'(s)}{B\rho}x = 0 \qquad \frac{d^2y}{ds^2} - \frac{B'(s)}{B\rho}y = 0$$

Combining with the previous slide

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = 0 \qquad \frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = 0$$

Hill's Equation

Define focusing strengths (with units of m^{-2})

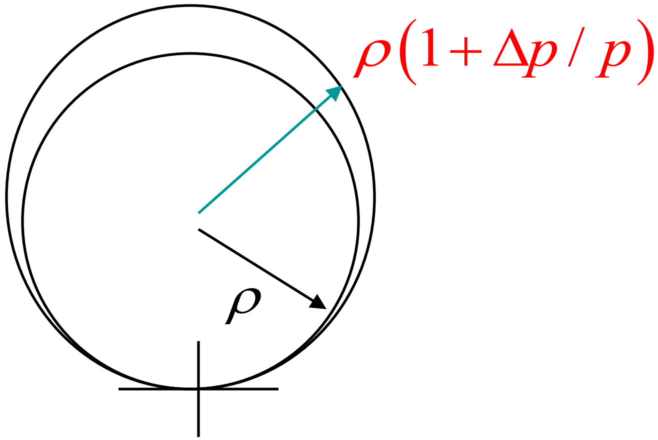
$$k_x(s) = \frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho} \qquad k_y = \frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}$$

$$\frac{d^2x}{ds^2} + k_x(s)x = 0 \qquad \frac{d^2y}{ds^2} + k_y(s)y = 0$$

Note that this is like the harmonic oscillator, or exponential for constant *K,* but more general in that the focusing strength, and hence oscillation frequency depends on *s*

Energy Effects

$$\Delta x(s) = \frac{p}{eB_y}\frac{\Delta p}{p}\left(1-\cos(s/\rho)\right)$$



This solution is not a solution to Hill's equation directly, but *is* a solution to the inhomogeneous Hill's Equations

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho_x^2(s)} + \frac{B'(s)}{B\rho}\right]x = \frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2} + \left[\frac{1}{\rho_y^2(s)} - \frac{B'(s)}{B\rho}\right]y = \frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

Inhomogeneous Hill's Equations

Fundamental transverse equations of motion in particle accelerators for small deviations from design trajectory

$$\frac{d^2x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]x=\frac{1}{\rho_x(s)}\frac{\Delta p}{p}$$

$$\frac{d^2y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]y=\frac{1}{\rho_y(s)}\frac{\Delta p}{p}$$

ρ radius of curvature for bends, B' transverse field gradient for magnets that focus (positive corresponds to horizontal focusing), $\Delta p/p$ momentum deviation from design momentum. Homogeneous equation is 2nd order *linear* ordinary differential equation.

Dispersion

From theory of linear ordinary differential equations, the general solution to the inhomogeneous equation is the sum of **any** solution to the inhomogeneous equation, called the particular integral, plus two linearly independent solutions to the homogeneous equation, whose amplitudes may be adjusted to account for boundary conditions on the problem.

$$x(s)=x_p(s)+A_x x_1(s)+B_x x_2(s) \qquad y(s)=y_p(s)+A_y y_1(s)+B_y y_2(s)$$

Because the inhomogeneous terms are proportional to $\Delta p/p$, the particular solution can generally be written as

$$x_p(s)=D_x(s)\frac{\Delta p}{p} \qquad y_p(s)=D_y(s)\frac{\Delta p}{p}$$

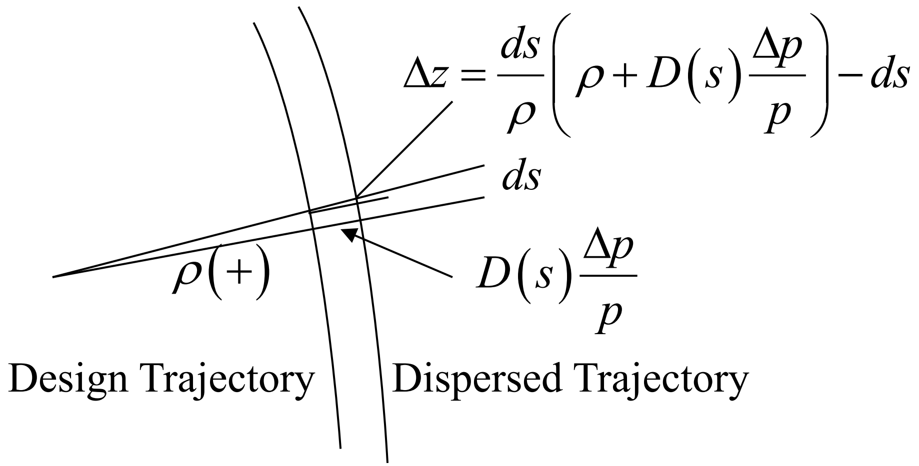
where the dispersion functions satisfy

$$\frac{d^2 D_x}{ds^2}+\left[\frac{1}{\rho_x^2(s)}+\frac{B'(s)}{B\rho}\right]D_x=\frac{1}{\rho_x(s)} \qquad \frac{d^2 D_y}{ds^2}+\left[\frac{1}{\rho_y^2(s)}-\frac{B'(s)}{B\rho}\right]D_y=\frac{1}{\rho_y(s)}$$

M_{56}

In addition to the transverse effects of the dispersion, there are important effects of the dispersion along the direction of motion. The primary effect is to change the time-of-arrival of the off-momentum particle compared to the on-momentum particle which traverses the design trajectory.

$$d(\Delta z) = D(s)\frac{\Delta p}{p}\frac{ds}{\rho(s)}$$



$$M_{56} = \int_{s_1}^{s_2} \left\{ \frac{D_x(s)}{\rho_x(s)} + \frac{D_y(s)}{\rho_y(s)} \right\} ds$$

Solutions Homogeneous Eqn.

Dipole

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left((s-s_i)/\rho\right) & \rho\sin\left((s-s_i)/\rho\right) \\ -\sin\left((s-s_i)/\rho\right)/\rho & \cos\left((s-s_i)/\rho\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} 1 & s-s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(s_i) \\ \dfrac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the focusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{k}\,(s-s_i)\right) & \sin\left(\sqrt{k}\,(s-s_i)\right)/\sqrt{k} \\ -\sqrt{k}\sin\left(\sqrt{k}\,(s-s_i)\right) & \cos\left(\sqrt{k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Quadrupole in the defocusing direction $k = B' / B\rho$

$$\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{-k}\,(s-s_i)\right) & \sinh\left(\sqrt{-k}\,(s-s_i)\right)/\sqrt{-k} \\ \sqrt{-k}\sinh\left(\sqrt{-k}\,(s-s_i)\right) & \cosh\left(\sqrt{-k}\,(s-s_i)\right) \end{pmatrix} \begin{pmatrix} x(s_i) \\ \frac{dx}{ds}(s_i) \end{pmatrix}$$

Transfer Matrices

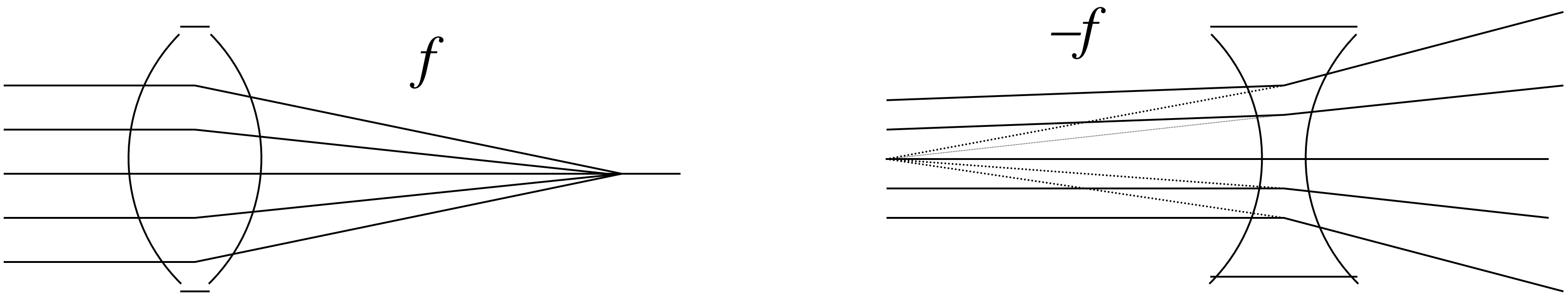
Dipole with bend Θ (put coordinate of final position in solution)

$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\Theta\right) & \rho\sin\left(\Theta\right) \\ -\sin\left(\Theta\right)/\rho & \cos\left(\Theta\right) \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Drift

$$\begin{pmatrix} x\left(s_{after}\right) \\ \dfrac{dx}{ds}\left(s_{after}\right) \end{pmatrix} = \begin{pmatrix} 1 & L_{drift} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{before}\right) \\ \dfrac{dx}{ds}\left(s_{before}\right) \end{pmatrix}$$

Thin Lenses

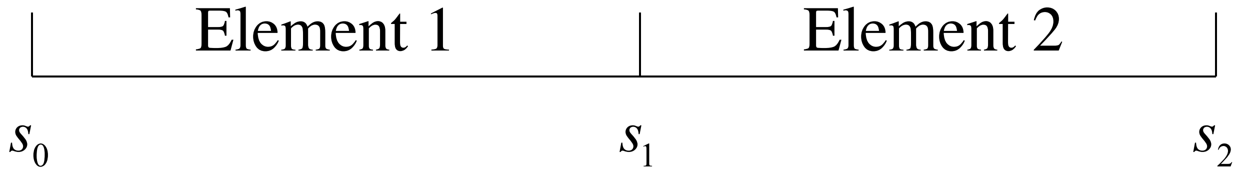


Thin Focusing Lens (limiting case when argument goes to zero!)

$$\begin{pmatrix} x\left(s_{lens}+\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}+\varepsilon\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x\left(s_{lens}-\varepsilon\right) \\ \dfrac{dx}{ds}\left(s_{lens}-\varepsilon\right) \end{pmatrix}$$

Thin Defocusing Lens: change sign of f

Composition Rule: Matrix Multiplication!



$$\begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix} = M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix} \qquad \begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = M_2 M_1 \begin{pmatrix} x(s_0) \\ x'(s_0) \end{pmatrix}$$

More generally

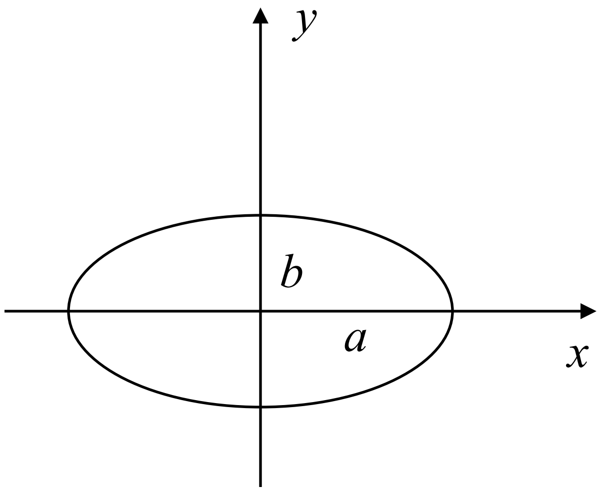
$$M_{tot} = M_N M_{N-1} \ldots M_2 M_1$$

Remember: First element farthest RIGHT

Some Geometry of Ellipses

Equation for an upright ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



In beam optics, the equations for ellipses are normalized (by multiplication of the ellipse equation by *ab*) so that the area of the ellipse divided by π appears on the RHS of the defining equation. For a general ellipse

$$Ax^2 + 2Bxy + Cy^2 = D$$

The area is easily computed to be

$$\frac{\text{Area}}{\pi} \equiv \varepsilon = \frac{D}{\sqrt{AC - B^2}} \qquad \text{Eqn. (1)}$$

So the equation is equivalently

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$$

$$\gamma = \frac{A}{\sqrt{AC - B^2}}, \quad \alpha = \frac{B}{\sqrt{AC - B^2}}, \quad \text{and} \quad \beta = \frac{C}{\sqrt{AC - B^2}}$$

When normalized in this manner, the equation coefficients clearly satisfy

$$\beta\gamma - \alpha^2 = 1$$

Example: the defining equation for the upright ellipse may be rewritten in following suggestive way

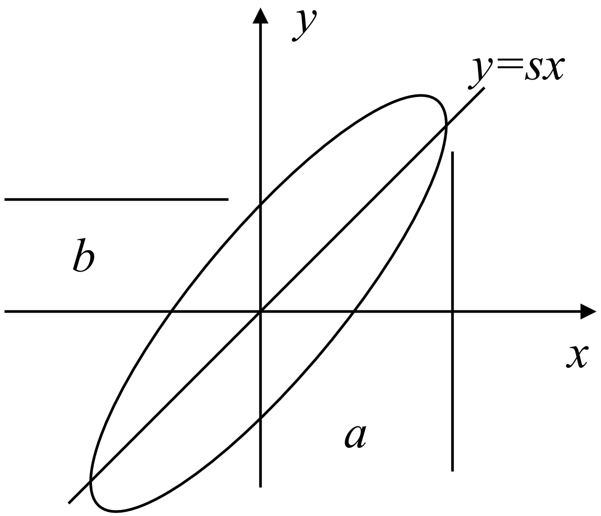
$$\frac{b}{a}x^2 + \frac{a}{b}y^2 = ab = \varepsilon$$

$\beta = a/b$ and $\gamma = b/a$, note $x_{\max} = a = \sqrt{\beta\varepsilon}$, $y_{\max} = b = \sqrt{\gamma\varepsilon}$

General Tilted Ellipse

Needs 3 parameters for a complete description. One way

$$\frac{b}{a}x^2 + \frac{a}{b}(y - sx)^2 = ab = \varepsilon$$



where s is a slope parameter, a is the maximum extent in the x-direction, and the y-intercept occurs at $\pm b$, and again ε is the area of the ellipse divided by π

$$\frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right)x^2 - 2s\frac{a}{b}xy + \frac{a}{b}y^2 = ab = \varepsilon$$

Identify

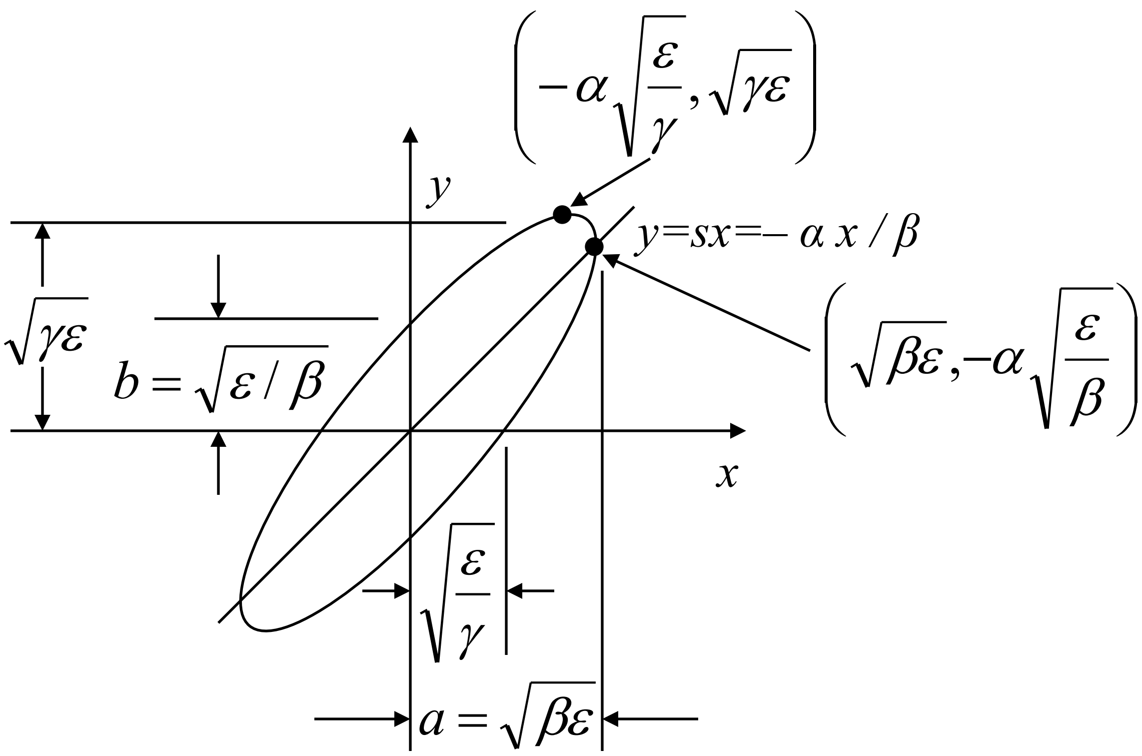
$$\gamma = \frac{b}{a}\left(1 + s^2\frac{a^2}{b^2}\right), \quad \alpha = -\frac{a}{b}s, \quad \beta = \frac{a}{b}$$

Note that $\beta\gamma - \alpha^2 = 1$ automatically, and that the equation for ellipse becomes

$$x^2 + \left(\beta y + \alpha x\right)^2 = \beta\varepsilon$$

by eliminating the (redundant!) parameter γ

Ellipse in the β-function Description



As for the upright ellipse $\quad x_{\max} = \sqrt{\beta\varepsilon}, \quad y_{\max} = \sqrt{\gamma\varepsilon}$

Area Theorem for Linear Optics

Under a general linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

an ellipse is transformed into another ellipse. Furthermore, if det (M) = 1, the area of the ellipse after the transformation is the same as that before the transformation.

Pf: Let the initial ellipse, normalized as above, be

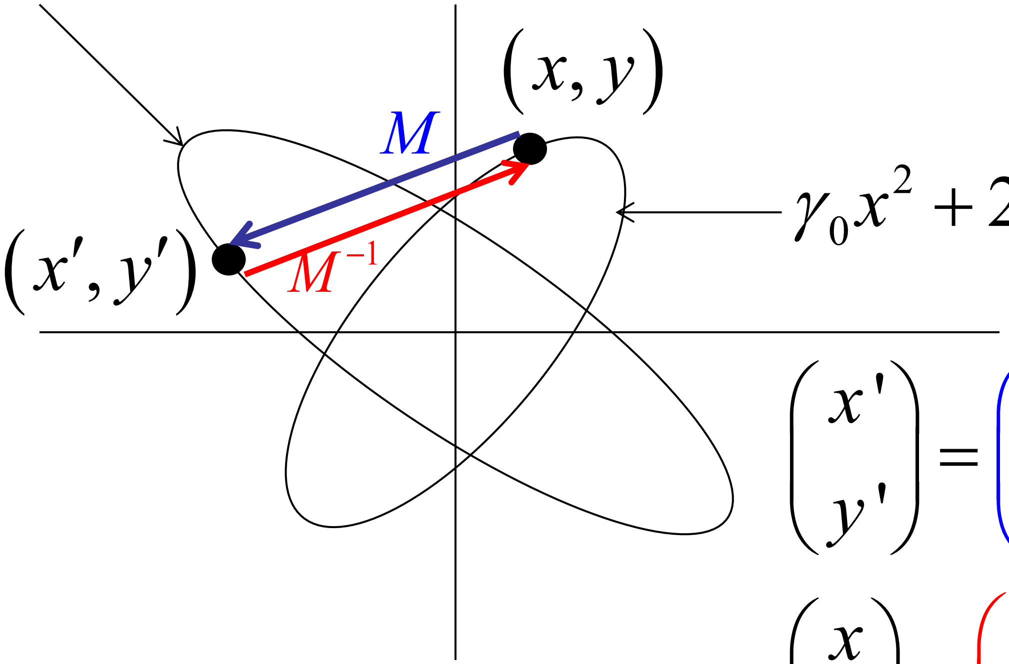
$$\gamma_0 x^2 + 2\alpha_0 xy + \beta_0 y^2 = \varepsilon_0$$

Effect of Transformation

Let the final ellipse be $\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon$

The transformed coordinates must solve this equation.

$$\gamma x'^2 + 2\alpha x'y' + \beta y'^2 = \varepsilon$$



$$\gamma_0 x^2 + 2\alpha_0 xy + \beta_0 y^2 = \varepsilon_0$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} M_{11}^{-1} & M_{12}^{-1} \\ M_{21}^{-1} & M_{22}^{-1} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The transformed coordinates must also solve the initial equation transformed.

Because

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(M^{-1}\right)_{11} & \left(M^{-1}\right)_{12} \\ \left(M^{-1}\right)_{21} & \left(M^{-1}\right)_{22} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

The transformed ellipse is

$$\gamma x'^2 + 2\alpha x'y' + \beta y'^2 = \varepsilon_0$$

$$\gamma = \left(M^{-1}\right)_{11}^2 \gamma_0 + 2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{21}\alpha_0 + \left(M^{-1}\right)_{21}^2 \beta_0$$

$$\alpha = \left(M^{-1}\right)_{11}\left(M^{-1}\right)_{12}\gamma_0 + \left(\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22} + \left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)\alpha_0 + \left(M^{-1}\right)_{21}\left(M^{-1}\right)_{22}\beta_0$$

$$\beta = \left(M^{-1}\right)_{12}^2 \gamma_0 + 2\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{22}\alpha_0 + \left(M^{-1}\right)_{22}^2 \beta_0$$

Because (verify!)

$$\beta\gamma-\alpha^2=\left(\beta_0\gamma_0-\alpha_0^2\right)$$

$$\times\left(\left(M^{-1}\right)_{21}^2\left(M^{-1}\right)_{12}^2+\left(M^{-1}\right)_{11}^2\left(M^{-1}\right)_{22}^2-2\left(M^{-1}\right)_{11}\left(M^{-1}\right)_{22}\left(M^{-1}\right)_{12}\left(M^{-1}\right)_{21}\right)$$

$$=\left(\beta_0\gamma_0-\alpha_0^2\right)\left(\det M^{-1}\right)^2$$

the area of the transformed ellipse (divided by π) is, by Eqn. (1)

$$\frac{\text{Area}}{\pi}=\varepsilon=\frac{\varepsilon_0}{\sqrt{\beta_0\gamma_0-\alpha_0^2}\left|\det M^{-1}\right|}=\varepsilon_0\,|\det M\,|$$

Tilted ellipse from the upright ellipse

In the tilted ellipse the y-coordinate is raised by the slope with respect to the un-tilted ellipse

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\gamma_0 = \frac{b}{a}, \qquad \alpha_0 = 0, \qquad \beta_0 = \frac{a}{b}, \qquad \left(M^{-1}\right)_{21} = -s$$

$$\therefore \quad \gamma = \frac{b}{a} + \frac{a}{b} s^2, \qquad \alpha = -\frac{a}{b} s, \qquad \beta = \frac{a}{b}$$

Because det (M)=1, the tilted ellipse has the same area as the upright ellipse, i.e., $\varepsilon = \varepsilon_0$.

Phase Advance of a Unimodular Matrix

Any two-by-two unimodular (Det (M) = 1) matrix with $|\mathrm{Tr}\, M| < 2$ can be written in the form

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

The *phase advance* of the matrix, μ, gives the eigenvalues of the matrix $\lambda = e^{\pm i\mu}$, and $\cos\mu = (\mathrm{Tr}\, M)/2$. Furthermore $\beta\gamma - \alpha^2 = 1$

Pf: The equation for the eigenvalues of M is

$$\lambda^2 - (M_{11} + M_{22})\lambda + 1 = 0$$

Because M is real, both λ and λ^* are solutions of the quadratic. Because

$$\lambda = \frac{\mathrm{Tr}(M)}{2} \pm i\sqrt{1-(\mathrm{Tr}(M)/2)^2}$$

For $|\mathrm{Tr}\ M| < 2$, $\lambda\lambda^* = 1$ and so $\lambda_{1,2} = e^{\pm i\mu}$. Consequently $\cos\mu = (\mathrm{Tr}\ M)/2$. Now the following matrix is trace-free.

$$M - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu) = \begin{pmatrix} \dfrac{M_{11}-M_{22}}{2} & M_{12} \\ M_{21} & \dfrac{M_{22}-M_{11}}{2} \end{pmatrix}$$

Simply choose

$$\alpha = \frac{M_{11} - M_{22}}{2\sin\mu}, \qquad \beta = \frac{M_{12}}{\sin\mu}, \qquad \gamma = -\frac{M_{21}}{\sin\mu}$$

and the sign of μ to properly match the individual matrix elements with $\beta > 0$. It is easily verified that $\beta\gamma - \alpha^2 = 1$. Now

$$M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(2\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(2\mu)$$

and more generally

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(n\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(n\mu)$$

Therefore, because sin and cos are both bounded functions, the matrix elements of any power of M remain bounded as long as $|\mathrm{Tr}\,(M)| < 2$.

NB, in some beam dynamics literature it is (incorrectly!) stated that the less stringent $|\mathrm{Tr}\,(M)| \leq 2$ ensures boundedness and/or stability. That equality cannot be allowed can be immediately demonstrated by counterexample. The upper triangular or lower triangular subgroups of the two-by-two unimodular matrices, i.e., matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

clearly have unbounded powers if $|x|$ is not equal to 0.

Significance of matrix parameters

Another way to interpret the parameters α, β, and γ, which represent the unimodular matrix M (these parameters are sometimes called the Twiss parameters or Twiss representation for the matrix) is as the "coordinates" of that specific set of ellipses that are mapped onto each other, or are invariant, under the linear action of the matrix. This result is demonstrated in

Thm: For the unimodular linear transformation

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

with $|\mathrm{Tr}\,(M)| < 2$, the ellipses

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

are invariant under the linear action of M, where c is any constant. Furthermore, these are the only invariant ellipses. Note that the theorem does not apply to $\pm I$, because $|\mathrm{Tr}\,(\pm I)| = 2$.

Pf: The inverse to M is clearly

$$M^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\mu) - \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin(\mu)$$

By the ellipse transformation formulas, for example

$$\begin{aligned} \beta' &= \beta^2 (\sin^2 \mu)\gamma + 2(-\beta \sin \mu)(\cos \mu + \alpha \sin \mu)\alpha + (\cos \mu + \alpha \sin \mu)^2 \beta \\ &= \beta \sin^2 \mu (1 + \alpha^2) - 2\beta\alpha^2 \sin^2 \mu + \beta \cos^2 \mu + \beta\alpha^2 \sin^2 \mu \\ &= (\sin^2 \mu + \cos^2 \mu)\beta = \beta \end{aligned}$$

Similar calculations demonstrate that $\alpha' = \alpha$ and $\gamma' = \gamma$. As det (M) = 1, $c' = c$, and therefore the ellipse is invariant. Conversely, suppose that an ellipse is invariant. By the ellipse transformation formula, the specific ellipse

$$\gamma_i x^2 + 2\alpha_i xy + \beta_i y^2 = \varepsilon$$

is invariant under the transformation by M only if

$$\begin{pmatrix}\gamma_i\\ \alpha_i\\ \beta_i\end{pmatrix} = \begin{pmatrix} (\cos\mu - \alpha\sin\mu)^2 & 2(\cos\mu - \alpha\sin\mu)(\gamma\sin\mu) & (\gamma\sin\mu)^2 \\ -(\cos\mu - \alpha\sin\mu)(\beta\sin\mu) & 1 - 2\beta\gamma\sin^2\mu & (\cos\mu + \alpha\sin\mu)(\gamma\sin\mu) \\ (\beta\sin\mu)^2 & -2(\cos\mu + \alpha\sin\mu)(\beta\sin\mu) & (\cos\mu + \alpha\sin\mu)^2 \end{pmatrix}\begin{pmatrix}\gamma_i\\ \alpha_i\\ \beta_i\end{pmatrix}$$

$$\equiv T_M \begin{pmatrix}\gamma_i\\ \alpha_i\\ \beta_i\end{pmatrix} \equiv T_M \vec{v},$$

i.e., if the vector $\vec{v}$ is *ANY* eigenvector of T_M with eigenvalue 1. All possible solutions may be obtained by investigating the eigenvalues and eigenvectors of T_M. Now

$$T_M \vec{v}_\lambda = \lambda \vec{v}_\lambda \quad \text{has a solution when Det} \left(T_M - \lambda I \right) = 0$$

i.e.,

$$\left(\lambda^2 + \left[2 - 4\cos^2 \mu \right] \lambda + 1 \right) \left(1 - \lambda \right) = 0$$

Therefore, *M* generates a transformation matrix T_M with at least one eigenvalue equal to 1. For there to be more than one solution with $\lambda = 1$,

$$1 + \left[2 - 4\cos^2 \mu \right] + 1 = 0, \quad \cos^2 \mu = 1, \quad \text{or} \quad M = \pm I$$

and we note that *all* ellipses are invariant when $M = \pm I$. But, these two cases are excluded by hypothesis. Therefore, M generates a transformation matrix T_M which always possesses a single nondegenerate eigenvalue 1; the set of eigenvectors corresponding to the eigenvalue 1, all proportional to each other, are the only vectors whose components (γ_i, α_i, β_i) yield equations for the invariant ellipses. For concreteness, compute that eigenvector with eigenvalue 1 normalized so $\beta_i\gamma_i - \alpha_i^2 = 1$

$$\vec{v}_{1,i} = \begin{pmatrix} \gamma_i \\ \alpha_i \\ \beta_i \end{pmatrix} = \beta \begin{pmatrix} -M_{21}/M_{12} \\ \left(M_{11} - M_{22}\right)/2M_{12} \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix}$$

All other eigenvectors with eigenvalue 1 have $\vec{v}_1 = \varepsilon \vec{v}_{1,i} / c$, for some value c.

Because Det (M) =1, the eigenvector $\vec{v}_{1,i}$ clearly yields the invariant ellipse

$$\gamma x^2 + 2\alpha xy + \beta y^2 = \varepsilon.$$

Likewise, the proportional eigenvector $\vec{v}_1$ generates the similar ellipse

$$\frac{\varepsilon}{c}\left(\gamma x^2 + 2\alpha xy + \beta y^2\right) = \varepsilon$$

Because we have enumerated all possible eigenvectors with eigenvalue 1, all ellipses invariant under the action of M, are of the form

$$\gamma x^2 + 2\alpha xy + \beta y^2 = c$$

To summarize, this theorem gives a way to tie the mathematical representation of a unimodular matrix in terms of its α, β, and γ, and its phase advance, to the equations of the ellipses invariant under the matrix transformation. The equations of the invariant ellipses when properly normalized have precisely the same α, β, and γ as in the Twiss representation of the matrix, but varying c.

Finally note that throughout this calculation c acts merely as a scale parameter for the ellipse. All ellipses similar to the starting ellipse, i.e., ellipses whose equations have the same α, β, and γ, but with different c, are also invariant under the action of M. Later, it will be shown that more generally

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + (\beta x' + \alpha x)^2\right)/\,\beta$$

is an invariant of the equations of transverse motion.

Applications to transverse beam optics

When the motion of particles in transverse phase space is considered, linear optics provides a good first approximation of the transverse particle motion. Beams of particles are represented by ellipses in phase space (i.e. in the (x, x') space). To the extent that the transverse forces are linear in the deviation of the particles from some pre-defined central orbit, the motion may analyzed by applying ellipse transformation techniques.

Transverse Optics Conventions: positions are measured in terms of length and angles are measured by radian measure. The area in phase space divided by π, ε, measured in m-rad, is called the emittance. In such applications, α has no units, β has units m/radian. Codes that calculate β, by widely accepted convention, drop the per radian when reporting results, it is implicit that the units for x' are radians.

Linear Transport Matrix

Within a linear optics description of transverse particle motion, the particle transverse coordinates at a location s along the beam line are described by a vector

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix}$$

If the differential equation giving the evolution of x is linear, one may define a linear transport matrix $M_{s',s}$ relating the coordinates at s' to those at s by

$$\begin{pmatrix} x(s') \\ \dfrac{dx}{ds}(s') \end{pmatrix} = M_{s',s} \begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix}$$

From the definitions, the concatenation rule $M_{s'',s} = M_{s'',s'}M_{s',s}$ must apply for all s' such that $s < s' < s''$ where the multiplication is the usual matrix multiplication.

Pf: The equations of motion, linear in x and dx/ds, generate a motion with

$$M_{s'',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} x(s'') \\ \frac{dx}{ds}(s'') \end{pmatrix} = M_{s'',s'}\begin{pmatrix} x(s') \\ \frac{dx}{ds}(s') \end{pmatrix} = M_{s'',s'}M_{s',s}\begin{pmatrix} x(s) \\ \frac{dx}{ds}(s) \end{pmatrix}$$

for all initial conditions ($x(s)$, $dx/ds(s)$), thus $M_{s'',s} = M_{s'',s'}M_{s',s}$.

Clearly $M_{s,s} = I$. As is shown next, the matrix $M_{s',s}$ is in general a member of the unimodular subgroup of the general linear group.

Ellipse Transformations From Hill's Equation

The equation governing the linear transverse dynamics in a particle accelerator, without acceleration, is *Hill's equation**

$$\frac{d^2x}{ds^2} + K(s)x = 0 \qquad \text{Eqn. (2)}$$

The transformation matrix taking a solution through an infinitesimal distance *ds* is

$$\begin{pmatrix} x(s+ds) \\ \dfrac{dx}{ds}(s+ds) \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{ds}{\text{rad}} \\ -K(s)ds\ \text{rad} & 1 \end{pmatrix} \begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} \equiv M_{s+ds,s} \begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix}$$

* Strictly speaking, Hill studied Eqn. (2) with periodic K. It was first applied to circular accelerators which had a periodicity given by the circumference of the machine. It is a now standard in the field of beam optics, to still refer to Eqn. 2 as Hill's equation, even in cases, as in linear accelerators, where there is no periodicity.

Suppose we are given the phase space ellipse

$$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \varepsilon$$

at location *s*, and we wish to calculate the ellipse parameters, after the motion generated by Hill's equation, at the location *s* + *ds*

$$\gamma(s+ds)x^2 + 2\alpha(s+ds)xx' + \beta(s+ds)x'^2 = \varepsilon'$$

Because, to order linear in *ds*, Det $M_{s+ds,s} = 1$, at all locations *s*, $\varepsilon' = \varepsilon$, and thus the phase space area of the ellipse after an infinitesimal displacement must equal the phase space area before the displacement. Because the transformation through a finite interval in *s* can be written as a series of infinitesimal displacement transformations, all of which preserve the phase space area of the transformed ellipse, we come to two important conclusions:

1. The phase space area is preserved after a finite integration of Hill's equation to obtain $M_{s',s}$, the transport matrix which can be used to take an ellipse at s to an ellipse at s'. This conclusion holds generally for all s' and s.

2. Therefore Det $M_{s',s} = 1$ for all s' and s, independent of the details of the functional form $K(s)$. (If desired, these two conclusions may be verified more analytically by showing that

$$\frac{d}{ds}\left(\beta\gamma-\alpha^2\right)=0 \quad\rightarrow\quad \beta(s)\gamma(s)-\alpha^2(s)=1, \quad \forall s$$

may be derived directly from Hill's equation.)

Evolution equations for α, β functions

The ellipse transformation formulas give, to order linear in ds

$$\beta(s+ds) = -2\alpha \frac{ds}{\text{rad}} + \beta(s)$$

$$\alpha(s+ds) = -\gamma(s)\frac{ds}{\text{rad}} + \alpha(s) + \beta(s) K ds \text{ rad}$$

So

$$\frac{d\beta}{ds}(s) = -\frac{2\alpha(s)}{\text{rad}}$$

$$\frac{d\alpha}{ds}(s) = \beta(s) K \text{ rad} - \frac{\gamma(s)}{\text{rad}}$$

Note that these two formulas are independent of the scale of the starting ellipse ε, and in theory may be integrated directly for $\beta(s)$ and $\alpha(s)$ given the focusing function $K(s)$. A somewhat easier approach to obtain $\beta(s)$ is to recall that the maximum extent of an ellipse, x_{max}, is $(\varepsilon\beta)^{1/2}(s)$, and to solve the differential equation describing its evolution. The above equations may be combined to give the following non-linear equation for $x_{max}(s) = w(s) = (\varepsilon\beta)^{1/2}(s)$

$$\frac{d^2w}{ds^2} + K(s)w = \frac{(\varepsilon/\text{rad})^2}{w^3}.$$

Such a differential equation describing the evolution of the maximum extent of an ellipse being transformed is known as an *envelope equation*.

It should be noted, for consistency, that the same $\beta(s) = w^2(s)/\varepsilon$ is obtained if one starts integrating the ellipse evolution equation from a different, but similar, starting ellipse. That this is so is an exercise.

The envelope equation may be solved with the correct boundary conditions, to obtain the β-function. α may then be obtained from the derivative of β, and γ by the usual normalization formula. Types of boundary conditions: Class I—periodic boundary conditions suitable for circular machines or periodic focusing lattices, Class II—initial condition boundary conditions suitable for linacs or recirculating machines.

Solution to Hill's Equation in Amplitude-Phase form

To get a more general expression for the phase advance, consider in more detail the single particle solutions to Hill's equation

$$\frac{d^2x}{ds^2} + K(s)x = 0$$

From the theory of linear ODEs, the general solution of Hill's equation can be written as the sum of the two linearly independent pseudo-harmonic functions

$$x(s) = Ax_+(s) + Bx_-(s)$$

where

$$x_\pm(s) = w(s)e^{\pm i\mu(s)}$$

are two particular solutions to Hill's equation, provided that

$$\frac{d^2w}{ds^2}+K(s)w=\frac{c^2}{w^3} \qquad \text{and} \qquad \frac{d\mu}{ds}(s)=\frac{c}{w^2(s)}, \quad \text{Eqns. (3)}$$

and where A, B, and c are constants (in s)

That specific solution with boundary conditions $x(s_1) = x_1$ and $dx/ds\ (s_1) = x'_1$ has

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} w(s_1)e^{i\mu(s_1)} & w(s_1)e^{-i\mu(s_1)} \\ \left[w'(s_1)+\frac{ic}{w(s_1)}\right]e^{i\mu(s_1)} & \left[w'(s_1)-\frac{ic}{w(s_1)}\right]e^{-i\mu(s_1)} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

Therefore, the unimodular transfer matrix taking the solution at $s = s_1$ to its coordinates at $s = s_2$ is

$$\begin{pmatrix} x_2 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \dfrac{w(s_2)}{w(s_1)}\cos\Delta\mu_{s_2,s_1} - \dfrac{w(s_2)w'(s_1)}{c}\sin\Delta\mu_{s_2,s_1} & \dfrac{w(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \\ -\dfrac{c}{w(s_2)w(s_1)}\left[1+\dfrac{w(s_2)w'(s_2)w(s_1)w'(s_1)}{c^2}\right]\sin\Delta\mu_{s_2,s_1} & \\ -\left[\dfrac{w'(s_1)}{w(s_2)}-\dfrac{w'(s_2)}{w(s_1)}\right]\cos\Delta\mu_{s_2,s_1} & \dfrac{w(s_1)}{w(s_2)}\cos\Delta\mu_{s_2,s_1} + \dfrac{w'(s_2)w(s_1)}{c}\sin\Delta\mu_{s_2,s_1} \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix}$$

where

$$\Delta\mu_{s_2,s_1} = \mu(s_2) - \mu(s_1) = \int_{s_1}^{s_2} \frac{c}{w^2(s)}ds$$

Case I: $K(s)$ periodic in s

Such boundary conditions, which may be used to describe circular or ring-like accelerators, or periodic focusing lattices, have $K(s + L) = K(s)$. L is either the machine circumference or period length of the focusing lattice.

It is natural to assume that there exists a unique periodic solution $w(s)$ to Eqn. (3a) when $K(s)$ is periodic. Here, we will assume this to be the case. Later, it will be shown how to construct the function explicitly. Clearly for w periodic

$$\phi(s) = \mu(s) - \mu_L s \qquad \text{with} \qquad \mu_L = \int_s^{s+L} \frac{c}{w^2(s)} ds$$

is also periodic by Eqn. (3b), and μ_L is independent of s.

The transfer matrix for a single period reduces to

$$\begin{pmatrix} \cos\mu_L - \frac{w(s)w'(s)}{c}\sin\mu_L & \frac{w^2(s)}{c}\sin\mu_L \\ -\frac{c}{w^2(s)}\left[1+\frac{w(s)w'(s)w(s)w'(s)}{c^2}\right]\sin\mu_L & \cos\mu_L + \frac{w'(s)w(s)}{c}\sin\mu_L \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos(\mu_L) + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin(\mu_L)$$

where the (now periodic!) matrix functions are

$$\alpha(s) = -\frac{w(s)w'(s)}{c}, \quad \beta(s) = \frac{w^2(s)}{c}, \quad \gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)}$$

By Thm. (2), these are the ellipse parameters of the periodically repeating, i.e., *matched* ellipses.

General formula for phase advance

In terms of the β-function, the phase advance for the period is

$$\mu_L = \int_0^L \frac{ds}{\beta(s)}$$

and more generally the phase advance between any two longitudinal locations s and s' is

$$\Delta\mu_{s',s} = \int_s^{s'} \frac{ds}{\beta(s)}$$

Transfer Matrix in terms of α and β

Also, the unimodular transfer matrix taking the solution from s to s' is

$$M_{s',s} = \begin{pmatrix} \sqrt{\dfrac{\beta(s')}{\beta(s)}}\left(\cos\Delta\mu_{s',s} + \alpha(s)\sin\Delta\mu_{s',s}\right) & \sqrt{\beta(s')\beta(s)}\sin\Delta\mu_{s',s} \\ -\dfrac{1}{\sqrt{\beta(s')\beta(s)}}\begin{bmatrix}\left(1+\alpha(s')\alpha(s)\right)\sin\Delta\mu_{s',s} \\ +\left(\alpha(s')-\alpha(s)\right)\cos\Delta\mu_{s',s}\end{bmatrix} & \sqrt{\dfrac{\beta(s)}{\beta(s')}}\left(\cos\Delta\mu_{s',s} - \alpha(s')\sin\Delta\mu_{s',s}\right) \end{pmatrix}$$

Note that this final transfer matrix and the final expression for the phase advance do not depend on the constant c. This conclusion might have been anticipated because different particular solutions to Hill's equation exist for all values of c, but from the theory of linear ordinary differential equations, the final motion is unique once x and dx/ds are specified somewhere.

Method to compute the β-function

Our previous work has indicated a method to compute the β-function (and thus w) directly, i.e., without solving the differential equation Eqn. (3). At a given location s, determine the one-period transfer map $M_{s+L,s}(s)$. From this find μ_L (which is independent of the location chosen!) from $\cos\mu_L = (M_{11}+M_{22}) / 2$, and by choosing the sign of μ_L so that $\beta(s) = M_{12}(s) / \sin\mu_L$ is positive. Likewise, $\alpha(s) = (M_{11}-M_{22}) / 2 \sin\mu_L$. Repeat this exercise at every location the β-function is desired.

By construction, the beta-function and the alpha-function, and hence w, are periodic because the single-period transfer map is periodic. It is straightforward to show $w=(c\beta(s))^{1/2}$ satisfies the envelope equation.

Courant-Snyder Invariant

Consider now a single particular solution of the equations of motion generated by Hill's equation. We've seen that once a particle is on an invariant ellipse for a period, it must stay on that ellipse throughout its motion. Because the phase space area of the single period invariant ellipse is preserved by the motion, the quantity that gives the phase space area of the invariant ellipse in terms of the single particle orbit must also be an invariant. This phase space area/π,

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2 = \left(x^2 + \left(\beta x' + \alpha x \right)^2 \right) / \beta$$

is called the Courant-Snyder invariant. It may be verified to be a constant by showing its derivative with respect to s is zero by Hill's equation, or by explicit substitution of the transfer matrix solution which begins at some initial value $s = 0$.

Pseudoharmonic Solution

$$\begin{pmatrix} x(s) \\ \dfrac{dx}{ds}(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}\left(\cos\Delta\mu_{s,0} + \alpha_0 \sin\Delta\mu_{s,0}\right) & \sqrt{\beta(s)\beta_0}\,\sin\Delta\mu_{s,0} \\ -\dfrac{1}{\sqrt{\beta(s)\beta_0}}\begin{bmatrix} (1+\alpha(s)\alpha_0)\sin\Delta\mu_{s,0} \\ +(\alpha(s)-\alpha_0)\cos\Delta\mu_{s,0} \end{bmatrix} & \sqrt{\dfrac{\beta_0}{\beta(s)}}\left(\cos\Delta\mu_{s,0} - \alpha(s)\sin\Delta\mu_{s,0}\right) \end{pmatrix} \begin{pmatrix} x_0 \\ \left.\dfrac{dx}{ds}\right|_0 \end{pmatrix}$$

gives

$$\left(x^2(s) + \left(\beta(s)x'(s) + \alpha(s)x(s)\right)^2\right)/\,\beta(s) = \left(x_0^2 + \left(\beta_0 x'_0 + \alpha_0 x_0\right)^2\right)/\,\beta_0 \equiv \varepsilon$$

Using the $x(s)$ equation above and the definition of ε, the solution may be written in the standard "pseudoharmonic" form

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\Delta\mu_{s,0} - \delta\right) \quad \text{where } \delta = \tan^{-1}\left(\frac{\beta_0 x'_0 + \alpha_0 x_0}{x_0}\right)$$

The the origin of the terminology "phase advance" is now obvious.

Floquet Transformation

Can define Floquet (sometimes called normalized) variables so that motion around the ellipse becomes motion along a unit circle

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s))$$

$$x'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\sin(\mu(s)) - \sqrt{\frac{\varepsilon}{\beta(s)}}\alpha(s)\cos(\mu(s))$$

$$x(s) = \sqrt{\varepsilon\beta(s)}\cos(\mu(s)) \qquad \beta(s)x'(s) + \alpha(s)x(s) = -\sqrt{\varepsilon\beta(s)}\sin(\mu(s))$$

$$\hat{x}(s) = \frac{x(s)}{\sqrt{\varepsilon\beta(s)}} \qquad \hat{y}(s) = \frac{\beta(s)x'(s) + \alpha(s)x(s)}{\sqrt{\varepsilon\beta(s)}}$$

$$\hat{x}^2(s) + \hat{y}^2(s) = 1$$

$$\hat{x}(0) = \cos\left(\mu(0)\right)$$

$$\hat{y}(0) = -\sin\left(\mu(0)\right)$$

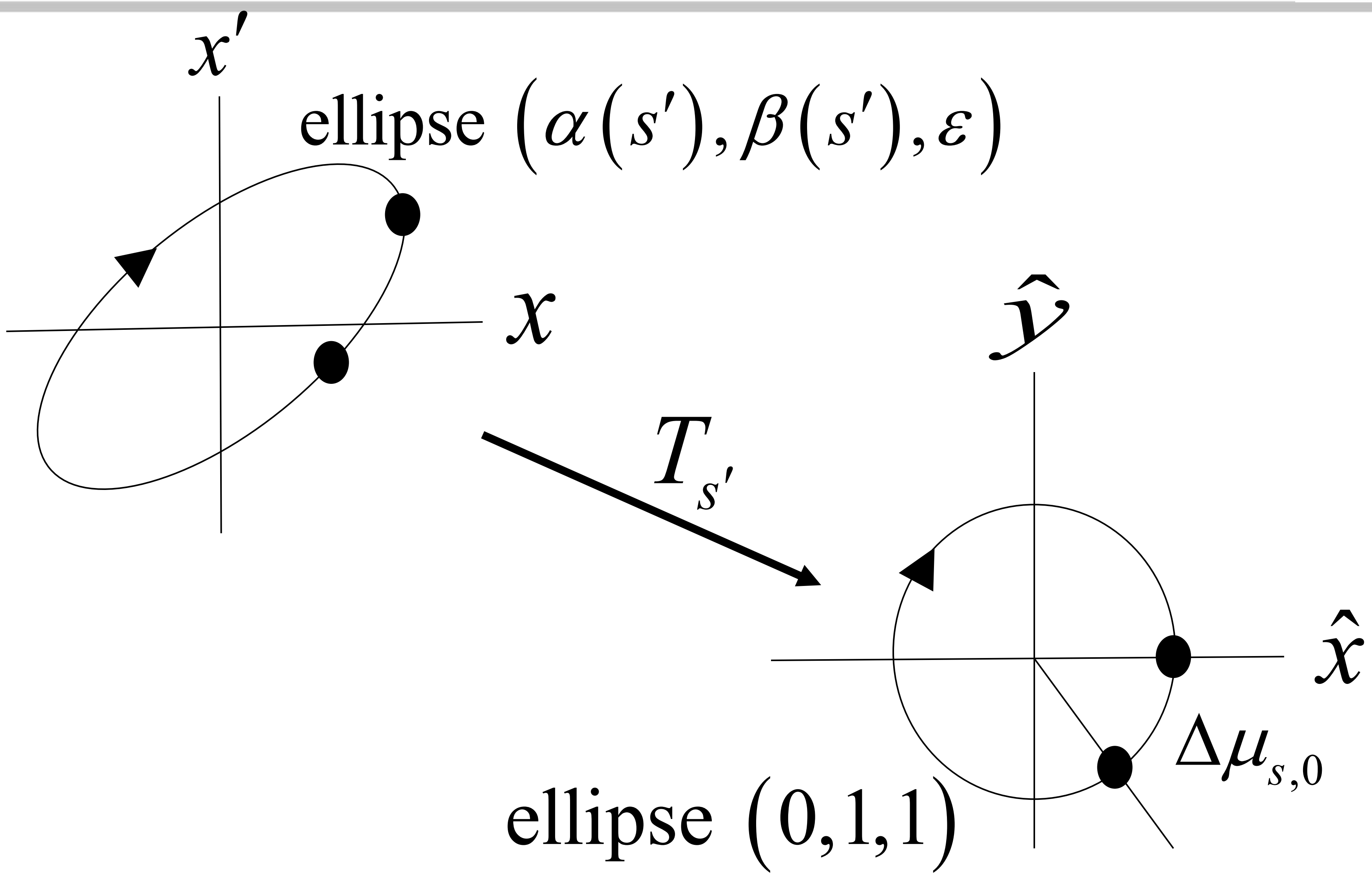
$$\begin{pmatrix} \hat{x}(s) \\ \hat{y}(s) \end{pmatrix} = \begin{pmatrix} \cos\Delta\mu_{s,0} & \sin\Delta\mu_{s,0} \\ -\sin\Delta\mu_{s,0} & \cos\Delta\mu_{s,0} \end{pmatrix} \begin{pmatrix} \hat{x}(0) \\ \hat{y}(0) \end{pmatrix} \qquad \Delta\mu_{s,0} = \mu(s) - \mu(0)$$

Clockwise rotation around the unit circle

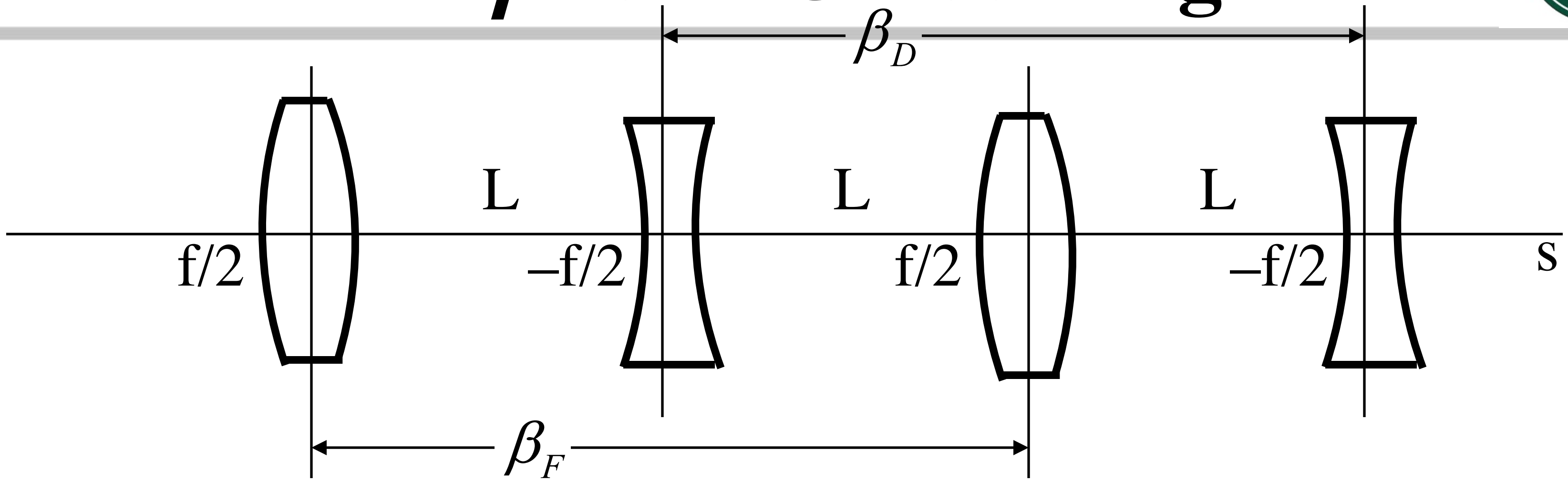
$$M_{s0} = T_s^{-1} \begin{pmatrix} \cos\Delta\mu_{s,0} & \sin\Delta\mu_{s,0} \\ -\sin\Delta\mu_{s,0} & \cos\Delta\mu_{s,0} \end{pmatrix} T_0$$

$$T_{s'} = \begin{pmatrix} 1/\sqrt{\varepsilon\beta(s')} & 0 \\ \alpha(s')/\sqrt{\varepsilon\beta(s')} & \sqrt{\beta(s')/\varepsilon} \end{pmatrix}$$

Effect of Transformation



α and β for AG Focusing



$$M_{s+2L,s} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1-2L^2/f^2 & 2\left(L+L^2/f\right) \\ -2\left(L/f^2-L^2/f^3\right) & 1-2L^2/f^2 \end{pmatrix}$$

$$\sin\mu/2 = f/L \qquad \alpha_F = 0 \qquad \beta_F = f\sqrt{\frac{1+L/f}{1-L/f}}$$

$$M_{s+2L,s} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/f & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} = \begin{pmatrix} 1-2L^2/f^2 & 2\left(L-L^2/f\right) \\ -2\left(L/f^2+L^2/f^3\right) & 1-2L^2/f^2 \end{pmatrix}$$

$$\sin\mu/2 = f/L \qquad \alpha_F = 0 \qquad \beta_D = f\sqrt{\frac{1-L/f}{1+L/f}}$$

Neck Tie Region of Stability

Now assume that the focal lengths are different, and possibly not alternating.

$$M_{s+2L,s}=\begin{pmatrix}1 & 0\\ -1/f_1 & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -2/f_2 & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1/f_1 & 1\end{pmatrix}=\begin{pmatrix}1-2L/f^* & 2L(1-L/f_2)\\ -\frac{2}{f^*}(1-L/f_1) & 1-2L/f^*\end{pmatrix}$$

$$\frac{1}{f^*}=\frac{1}{f_1}+\frac{1}{f_2}-\frac{L}{f_1 f_2}$$

$$|\mathrm{Tr}(\mathrm{M})|<2\rightarrow$$

$$0<L/f^*<1$$

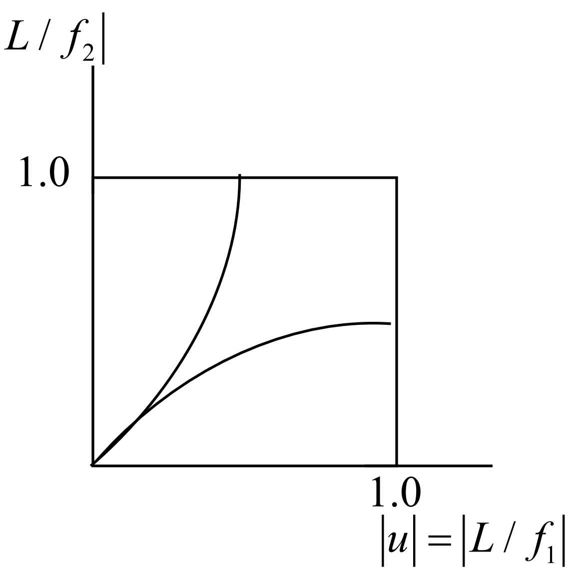
$$0<u+v-uv<1$$

$$0<-u-v-uv<1$$

$$0<-uv<1$$

$$|u|,|v|<1$$

$$|u|=\frac{|v|}{1+|v|},|v|=\frac{|u|}{1+|u|}$$



Case II: $K(s)$ not periodic

In a linac or a recirculating linac there is no closed orbit or natural machine periodicity. Designing the transverse optics consists of arranging a focusing lattice that assures the beam particles coming into the front end of the accelerator are accelerated (and sometimes decelerated!) with as small beam loss as is possible. Therefore, it is imperative to know the initial beam phase space injected into the accelerator, in addition to the transfer matrices of all the elements making up the focusing lattice of the machine. An initial ellipse, or a set of initial conditions that somehow bound the phase space of the injected beam, are tracked through the acceleration system element by element to determine the transmission of the beam through the accelerator. The designs are usually made up of well-understood "modules" that yield known and understood transverse beam optical properties.

Definition of β function

Now the pseudoharmonic solution applies even when $K(s)$ is not periodic. Suppose there is an ellipse, the design injected ellipse, which tightly includes the phase space of the beam at injection to the accelerator. Let the ellipse parameters for this ellipse be α_0, β_0, and γ_0. A function $\beta(s)$ is simply defined by the ellipse transformation rule

$$\beta(s) = (M_{12}(s))^2 \gamma_0 - 2M_{12}(s)M_{11}(s)\alpha_0 + (M_{11}(s))^2 \beta_0$$
$$= \left[(M_{12}(s))^2 + (\beta_0 M_{11}(s) - \alpha_0 M_{12}(s))^2\right] / \beta_0$$

where

$$M_{s,0} \equiv \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$

One might think to evaluate the phase advance by integrating the beta-function. Generally, it is far easier to evaluate the phase advance using the general formula,

$$\tan \Delta\mu_{s',s} = \frac{\left(M_{s',s}\right)_{12}}{\beta(s)\left(M_{s',s}\right)_{11} - \alpha(s)\left(M_{s',s}\right)_{12}}$$

where $\beta(s)$ and $\alpha(s)$ are the ellipse functions at the entrance of the region described by transport matrix $M_{s',s}$. Applied to the situation at hand yields

$$\tan \Delta\mu_{s,0} = \frac{M_{12}(s)}{\beta_0 M_{11}(s) - \alpha_0 M_{12}(s)}$$

Beam Matching

Fundamentally, in circular accelerators beam matching is applied in order to guarantee that the beam envelope of the real accelerator beam does not depend on time. This requirement is one part of the definition of having a stable beam. With periodic boundary conditions, this means making beam density contours in phase space align with the invariant ellipses (in particular at the injection location!) given by the ellipse functions. Once the particles are on the invariant ellipses they stay there (in the linear approximation!), and the density is preserved because the single particle motion is around the invariant ellipses. In linacs and recirculating linacs, usually different purposes are to be achieved. If there are regions with periodic focusing lattices within the linacs, matching as above ensures that the beam envelope does not grow going down the lattice. Sometimes it is advantageous to have specific values of the ellipse functions at specific longitudinal locations. Other times, re/matching is done to preserve the beam envelopes of a good beam solution as changes in the lattice are made to achieve other purposes, e.g. changing the dispersion function or changing the chromaticity of regions where there are bends (see the next chapter for definitions). At a minimum, there is usually a matching done in the first parts of the injector, to take the phase space that is generated by the particle source, and change this phase space in a way towards agreement with the nominal transverse focusing design of the rest of the accelerator. The ellipse transformation formulas, solved by computer, are essential for performing this process.

Dispersion Calculation

Begin with the inhomogeneous Hill's equation for the dispersion.

$$\frac{d^2 D}{ds^2} + K(s) D = \frac{1}{\rho(s)}$$

Write the general solution to the inhomogeneous equation for the dispersion as before.

$$D(s) = D_p(s) + A x_1(s) + B x_2(s)$$

$$x_1(s) = M_{s,s_1;1,1} D(s_1) \qquad x_1'(s_1) = M_{s,s_1;2,1} D(s_1)$$

$$x_2(s) = M_{s,s_1;1,2} D'(s_1) \qquad x_2'(s_1) = M_{s,s_1;2,2} D'(s_1)$$

Here D_p can be any particular solution, and we suppose that the dispersion and it's derivative are known at the location s_1, and we wish to determine their values at s. x_1 and x_2 are linearly independent solutions to the homogeneous differential equation because they are transported by the transfer matrix solution $M_{s,s1}$ already found.

To build up the general solution, choose that particular solution of the inhomogeneous equation with homogeneous boundary conditions

$$D_{p,0}(s_1) = D'_{p,0}(s_1) = 0$$

Evaluate A and B by the requirement that the dispersion and it's derivative have the proper value at s_1 (x_1 and x_2 need to be linearly independent!)

$$M_{s1,s1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow A = B = 1$$

$$D(s_2) = D_{p,0}(s_2 - s_1) + \left(M_{s_2,s_1}\right)_{11} D(s_1) + \left(M_{s_2,s_1}\right)_{12} D'(s_1)$$

$$D'(s_2) = D'_{p,0}(s_2 - s_1) + \left(M_{s_2,s_1}\right)_{21} D(s_1) + \left(M_{s_2,s_1}\right)_{22} D'(s_1)$$

3 by 3 Matrices for Dispersion Tracking

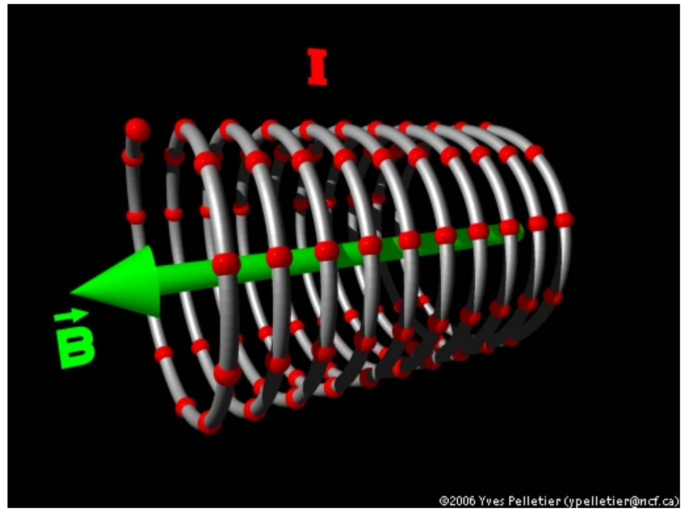
$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} \left(M_{s_2,s_1}\right)_{11} & \left(M_{s_2,s_1}\right)_{12} & D_{p,0}(s_2 - s_1) \\ \left(M_{s_2,s_1}\right)_{21} & \left(M_{s_2,s_1}\right)_{22} & D'_{p,0}(s_2 - s_1) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}$$

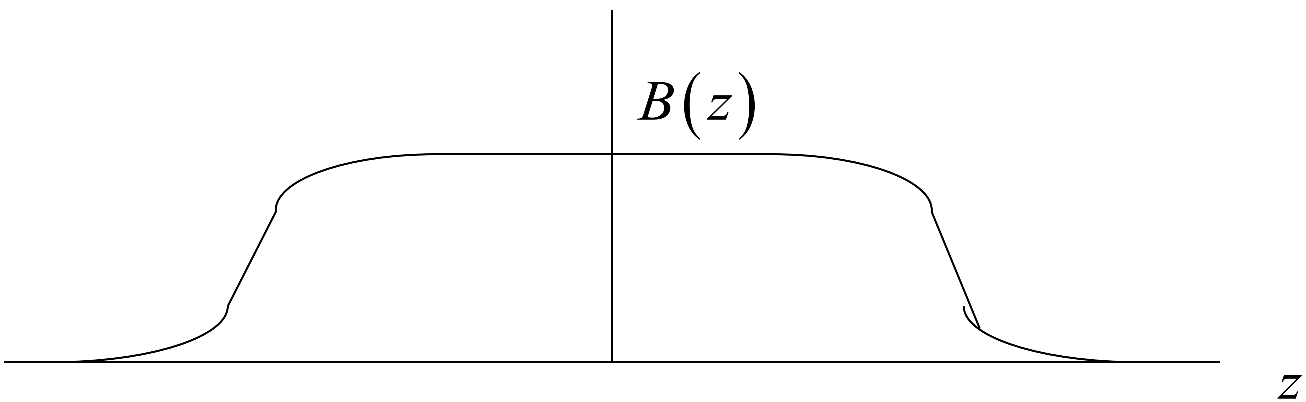
Particular solutions to inhomogeneous equation for constant K and constant ρ and vanishing dispersion and derivative at $s = 0$

	$K < 0$	$K = 0$	$K > 0$
$D_{p,0}(s)$	$\frac{1}{\lvert K\rvert \rho}\left(\cosh\left(\sqrt{\lvert K\rvert}s\right) - 1\right)$	$\frac{s^2}{2\rho}$	$\frac{1}{K\rho}\left(1 - \cos\left(\sqrt{K}s\right)\right)$
$D'_{p,0}(s)$	$\frac{1}{\sqrt{\lvert K\rvert}\rho}\sinh\left(\sqrt{\lvert K\rvert}s\right)$	$\frac{s}{\rho}$	$\frac{1}{\sqrt{K}\rho}\sin\left(\sqrt{K}s\right)$

Solenoid Focussing

Can also have continuous focusing in both transverse directions by applying solenoid magnets:





Busch's Theorem

For cylindrical symmetry magnetic field described by a vector potential:

$$\vec{A} = A_\theta(z,r)\hat{\theta} \qquad B_z = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta(z,r)\right) \quad \text{is nearly constant}$$

$$\therefore A_\theta(z,r) \doteq \frac{B_z(r=0,z)r}{2} \qquad\qquad B_r = \frac{B_z'(r=0,z)r}{2}$$

Conservation of Canonical Momentum gives Busch's Theorem:

$$P_\theta = \gamma m r^2 \dot{\theta} + qrA_\theta = const$$

for particle with $\dot{\theta} = 0$ where $B_z = 0,\ P_\theta = 0$

$$\dot{\theta} = -\frac{qB_z}{2\gamma m} = -\frac{\Omega_c}{2} = -\omega_{Larmor}$$

Beam rotates at the Larmor frequency which implies coupling between horizontal and vertical dimensions

Radial Equation

$$\frac{d}{dt}\left(\gamma m\dot{r}\right)-\gamma m r\omega_L^2=qr\dot{\theta}B_z=-2\gamma m r\omega_L^2$$

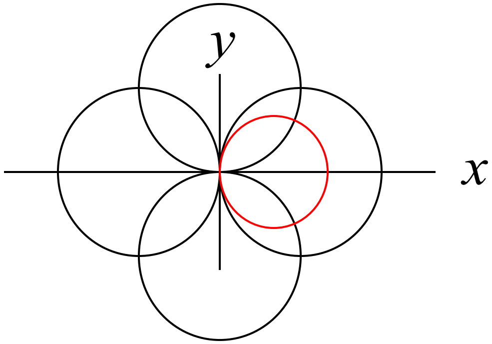
$$\therefore k=\frac{\omega_L^2}{\beta_z^2c^2}$$

thin lens focal length

$$\frac{1}{f}=\frac{e^2\int_{-\infty}^{\infty}B_z^2dz}{4\beta_z^2\gamma^2m^2c^2}$$

weak compared to quadrupole for high γ

If go to full ¼ oscillation inside the magnetic field in the "thick" lens case, all particles end up at r = 0! Non-zero emittance spreads out perfect focusing!



Larmor's Theorem

This result is a special case of a more general result. If go to frame that rotates with the local value of Larmor's frequency, then the transverse dynamics including the magnetic field are simply those of a harmonic oscillator with frequency equal to the Larmor frequency. Any force from the magnetic field linear in the field strength is "transformed away" in the Larmor frame. And the motion in the two transverse degrees of freedom is now decoupled. Pf: The equations of motion are

$$\frac{d}{dt}(\gamma m\dot{r})-\gamma mr\dot{\theta}^2=qr\dot{\theta}B_z$$

$$\gamma mr^2\dot{\theta}+qA_\theta=cons=P_\theta$$

$$\frac{d}{dt}(\gamma m\dot{r})-\gamma mr\dot{\theta}'^2+2\gamma mr\dot{\theta}'\omega_L-\gamma mr\omega_L^2=qr\dot{\theta}'B_z-qr\omega_L B_z$$

$$\gamma mr^2\dot{\theta}'=P_\theta$$

$$\left.\begin{aligned}\frac{d}{dt}(\gamma m\dot{r})-\gamma mr\dot{\theta}'^2&=-\gamma mr\omega_L^2\\ \gamma mr^2\dot{\theta}'&=P_\theta\end{aligned}\right\}\text{ 2-D Harmonic Oscillator}$$