Magnetic Multipoles, Magnet Design

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Magnetic Multipoles – Outline

- Solutions to Maxwell's equations for magneto static fields:
  - in two dimensions (multipole fields)
  - in three dimensions (fringe fields, end effects, insertion devices...)
- How to construct multipole fields in two dimensions, using electric currents and magnetic materials, considering idealized situations.

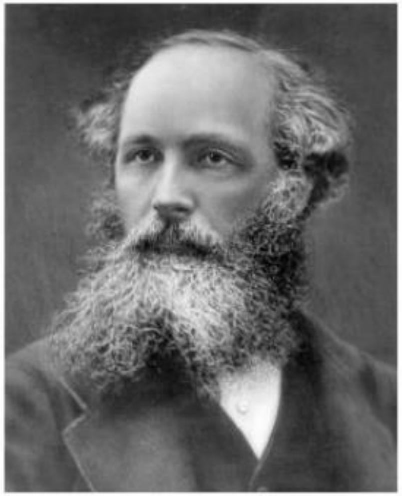
  - A. Wolski, University of Liverpool and the Cockcroft Institute, CAS Specialized Course on Magnets, 2009, http://cas.web.cern.ch/cas/Belgium-2009/Lectures/PDFs/Wolski-1.pdf

Basis

- Vector calculus in Cartesian and polar coordinate systems;
- Stokes' and Gauss' theorems
- Maxwell's equations and their physical significance
- Types of magnets commonly used in accelerators.

  - following notation used in: A. Chao and M. Tigner, "Handbook of Accelerator Physics and Engineering," World Scientific (1999).

Maxwell's Equations



James Clerk Maxwell
(1831-1879)

* MKS unit

$\nabla \cdot \vec{D} = \rho$	Gauss's law for electric field
$\nabla \cdot \vec{B} = 0$	Gauss's law for magnetic field
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	Ampere's law

$$\begin{aligned} \vec{D} &= \varepsilon \vec{E}, \\ \vec{B} &= \mu \vec{H}. \end{aligned}$$

$\vec{D}$	electric displacement (C/m^2)
$\vec{E}$	electric field (V/m)
$\vec{B}$	magnetic field $(T$ or $V \cdot s/m^2)$
$\vec{H}$	magnetic field intensity (A/m)
ρ	volume charge density (C/m^3)
$\vec{J}$	current density (A/m^2)

electric permittivity ε and magnetic permeability μ

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*
$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$\nabla \cdot$ (div)

$\nabla \times$ (curl)

Maxwells Equations for Magnets

In these lectures, I shall consider only magnetostatic fields.

Maxwell's equations for the magnetic field become:

$$\nabla \cdot \vec{B} = 0, \quad (1)$$

$$\nabla \times \vec{H} = \vec{J}. \quad (2)$$

In this first lecture, we shall show that multipole fields provide solutions to these equations in two dimensions, i.e. where the fields and currents are independent of one coordinate (z). We shall also deduce the current distributions and material property and geometries that can generate fields with specified multipole components.

Physical Interpretation of $\nabla \cdot \vec{B} = 0$

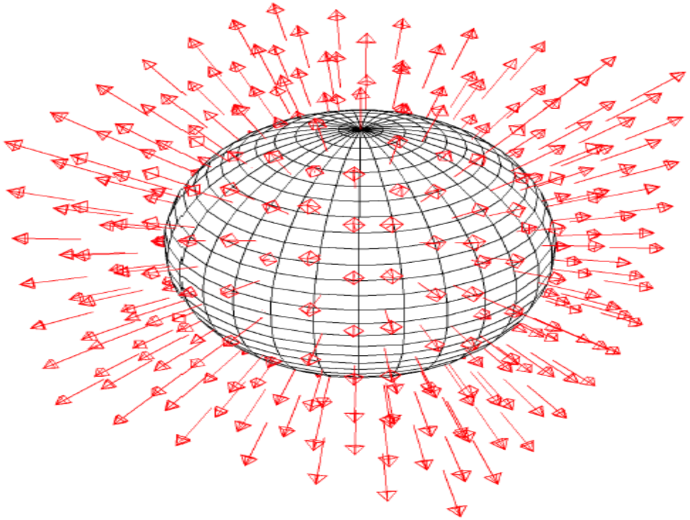
Gauss' theorem tells us that for *any* smooth vector field $\vec{B}$:

$$\int_V \nabla \cdot \vec{B}\, dV = \oint_S \vec{B} \cdot d\vec{S},$$

where the closed surface S bounds the region V.

Applied to Maxwell's equation $\nabla \cdot \vec{B} = 0$, Gauss' theorem tells us that the total flux entering a bounded region equals the total flux leaving the same region.

Johann Carl Gauss
(1777-1855)



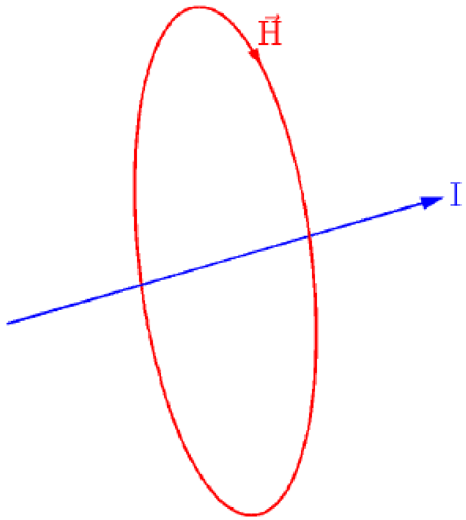
Stokes' theorem tells us that for *any* smooth vector field $\vec{H}$:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell}, \qquad (4)$$

where the closed loop C bounds the surface S.

Applied to Maxwell's equation $\nabla \times \vec{H} = \vec{J}$, Stokes' theorem tells us that the magnetic field $\vec{H}$ integrated around a closed loop equals the total current passing through that loop:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} = I. \qquad (5)$$



Linearity and Superposition

Maxwell's equations are *linear*:

$$\nabla \cdot \left(\vec{B}_1 + \vec{B}_2 \right) = \nabla \cdot \vec{B}_1 + \nabla \cdot \vec{B}_2, \tag{6}$$

and:

$$\nabla \times \left(\vec{H}_1 + \vec{H}_2 \right) = \nabla \times \vec{H}_1 + \nabla \times \vec{H}_2. \tag{7}$$

This means that if two fields $\vec{B}_1$ and $\vec{B}_2$ satisfy Maxwell's equations, so does their sum $\vec{B}_1 + \vec{B}_2$.

As a result, we can apply the *principle of superposition* to construct complicated magnetic fields just by adding together a set of simpler fields.

Multipole Fields

Let us first consider fields that satisfy Maxwell's equations in free space, e.g. the interior of an accelerator vacuum chamber. Here, we have $\vec{J} = 0$, and $\vec{B} = \mu_0 \vec{H}$; hence, Maxwell's equations (1) and (2) become:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{B} = 0. \tag{8}$$

Consider the field given by $B_z =$ constant, and:

$$B_y + iB_x = C_n \left(x + iy\right)^{n-1}, \tag{9}$$

where n is a positive integer, and C_n is a complex number.

Note that the field components B_x, B_y and B_z are all *real*; we are only using complex numbers for convenience.

Multipole Fields

Now consider the differential operator:

$$\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}. \tag{10}$$

Applying this operator to the left hand side of (9) gives:

$$\begin{aligned}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(B_y + iB_x) &= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) + i\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right), \\ &= \left[\nabla \times \vec{B}\right]_z + i\nabla \cdot \vec{B}. \end{aligned} \tag{11}$$

In the final step, we have used the fact that B_z is constant. Also using this fact, and the fact that B_x and B_y are independent of z, we see that the x and y components of $\nabla \times \vec{B}$ vanish.

Applying the operator (10) to the right hand side of (9) gives:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy)^{n-1} = (n-1)\,(x+iy)^{n-2} + i^2(n-1)\,(x+iy)^{n-2} = 0. \tag{12}$$

Multipole Fields

Hence, applying the operator (10) to both sides of equation (9), we find that:

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{B} = 0. \tag{13}$$

Therefore, the field (9) satisfies Maxwell's equations for a magnetostatic system in free space.

Of course, this analysis simply tells us that the field (9):

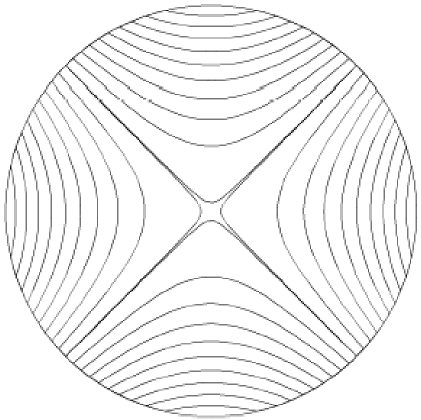
$$\boxed{B_y + iB_x = C_n \left(x + iy\right)^{n-1}}$$

is a possible solution to Maxwell's equations in the situation we have described: it does not tell us how to generate such a field.

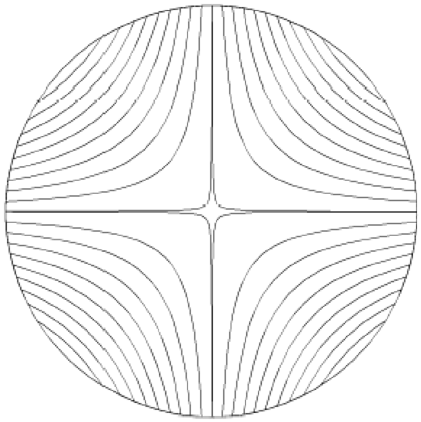
Fields given by (9) are called *multipole fields*. Note that, since Maxwell's equations are linear, we can superpose any number of multipole fields, and obtain a valid solution to Maxwell's equations.

Multipole Fields

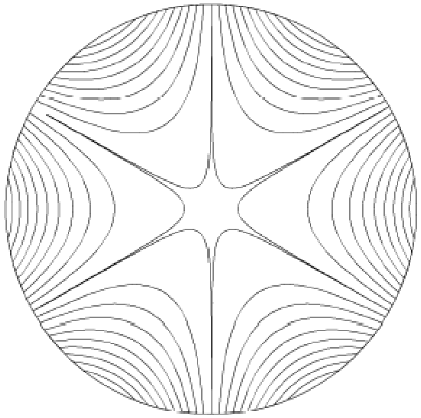
C_2 =real, normal quadrupole



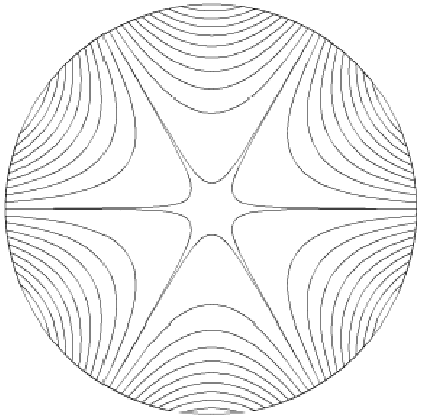
C_2 =imaginary, skew quadrupole



C_3 =real, normal sextupole



C_3 =imaginary, skew sextupole



Multipole Fields

For $C_n = 0$ for all n, we have:

$$B_x = B_y = 0, \qquad B_z = \text{constant.} \tag{14}$$

This is a solenoid field, and is not generally regarded as a multipole field.

In the conventional notation (see Chao and Tigner), we rewrite the field (9) as:

$$B_y + iB_x = B_{\text{ref}} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{R_{\text{ref}}} \right)^{n-1} . \tag{15}$$

The b_n are the "normal multipole coefficients", and the a_n are the "skew multipole coefficients". B_{ref} and R_{ref} are a reference field strength and a reference radius, whose values may be chosen arbitrarily; however their values will affect the values of the multipole coefficients for a given field.

Multipole Fields

The interpretation of the multipole coefficients is probably best understood by considering the field behaviour in the plane $y = 0$:

$$B_y = B_{\text{ref}} \sum_{n=1}^{\infty} b_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}, \quad \text{and} \quad B_x = B_{\text{ref}} \sum_{n=1}^{\infty} a_n \left(\frac{x}{R_{\text{ref}}} \right)^{n-1}. \tag{16}$$

A single multipole component with $n = 1$ is a dipole field: $B_y = b_1 B_{\text{ref}}$ is constant, and $B_x = a_1 B_{\text{ref}}$ is also constant.

A single multipole component with $n = 2$ is a quadrupole field:

$$B_y = b_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}, \quad \text{and} \quad B_x = a_2 B_{\text{ref}} \frac{x}{R_{\text{ref}}}. \tag{17}$$

Both B_y and B_x vary linearly with x.

For $n = 3$ (sextupole), the field components vary as x^2, etc.

Multipole Fields from a Current Distribution

To see how to generate a multipole field, we start with the magnetic field around a thin wire carrying a current I_0. Generally, the magnetic field in the presence of a current density $\vec{J}$ is given by Maxwell's equation (2):

$$\nabla \times \vec{H} = \vec{J}.$$

Consider a thin straight wire of infinite length, oriented along the z axis. Let us integrate Maxwell's equation (2) over a circular disc of radius r centered on the wire, and normal to the wire:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I_0, \tag{18}$$

where we have used the fact that the integral of the current density over the cross section of the wire equals the total current flowing in the wire.

Multipole Fields from a Current Distribution

Now we apply Stokes' theorem, which tells us that for *any* smooth vector field F:

$$\int_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{\ell}, \tag{19}$$

where C is the closed curve bounding the surface S.

Applied to equation (18), Stokes' theorem gives us:

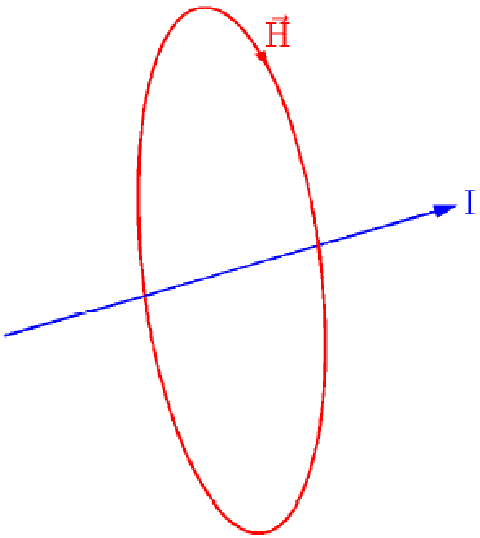
$$\oint_C \vec{H} \cdot d\vec{\ell} = I_0, \tag{20}$$

By symmetry, the magnetic field must be the same magnitude at equal distances from the wire. We also know, from Gauss' theorem applied to $\nabla \cdot \vec{B} = 0$, that there can be no radial component to the magnetic field.

Multipole Fields from a Current Distribution

Hence, the magnetic field at any point is tangential to a circle centered on the wire and passing through that point. We also find, by performing the integral in (20), that the magnitude of the magnetic field at distance r from the wire is given by:

$$\vec{H} = \frac{I_0}{2\pi r}. \tag{21}$$



If there are no magnetic materials present, $\mu = \mu_0$, so:

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I_0}{2\pi r}. \tag{22}$$

Multipole Fields from a Current Distribution

Now, let us work out the field at a point $\vec{r} = (x, y, z)$ from a current parallel to the z axis, but displaced from it. The line of current is defined by $x = x_0$, $y = y_0$.

The magnitude of the field is given, from (22) by:

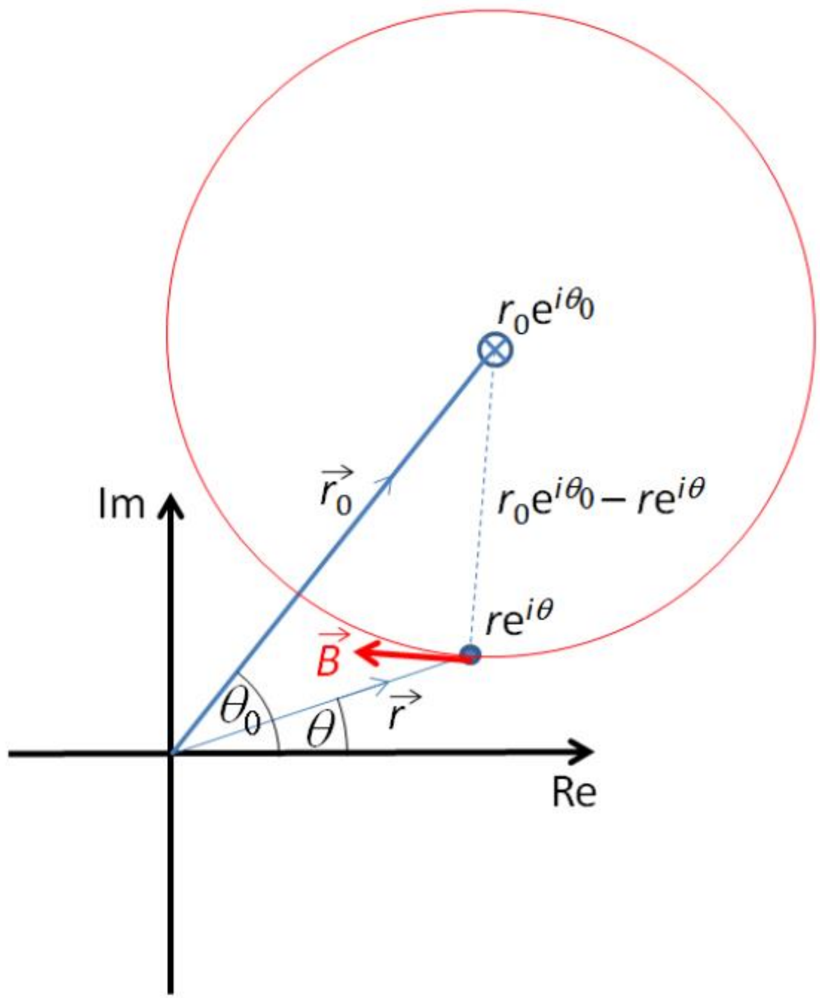
$$B = \frac{\mu_0 I_0}{2\pi \left| \vec{r} - \vec{r}_0 \right|},$$

where the vector $\vec{r}_0$ has components $\vec{r}_0 = (x_0, y_0, z)$.

Since the field at $\vec{r}$ is perpendicular to $\vec{r} - \vec{r}_0$, the field vector is given by:

$$\vec{B} = \frac{\mu_0 I_0}{2\pi} \frac{(y - y_0, -x + x_0, 0)}{\left| \vec{r} - \vec{r}_0 \right|^2}.$$

Multipole Fields from a Current Distribution



Multipole Fields from a Current Distribution

$$\vec{B} = \frac{\mu_0 I_0}{2\pi} \frac{(y - y_0, -x + x_0, 0)}{|\vec{r} - \vec{r}_0|^2}. \qquad (24)$$

It is convenient to express the field (24) in complex notatio
Writing:

$$x + iy = re^{i\theta}, \quad \text{and} \quad x_0 + iy_0 = r_0 e^{i\theta_0}, \qquad (25)$$

we find that:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi} \frac{\left(r_0 e^{-i\theta_0} - re^{-i\theta}\right)}{\left|r_0 e^{i\theta_0} - re^{i\theta}\right|^2}. \qquad (26)$$

Multipole Fields from a Current Distribution

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi} \frac{\cancel{\left(r_0 e^{-i\theta_0} - r e^{-i\theta}\right)}}{\left| r_0 e^{i\theta_0} - r e^{i\theta} \right|^{\cancel{2}}}. \qquad (26)$$

Using the fact that for any complex number ζ, we have $|\zeta|^2 = \zeta\zeta^*$:

$$\left(r_0 e^{i\theta_0} - r e^{i\theta} \right)^*$$

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_0}{2\pi} \frac{1}{r_0 e^{i\theta_0} - r e^{i\theta}} \\ &= \frac{\mu_0 I_0}{2\pi r_0} \frac{e^{-i\theta_0}}{\left(1 - \frac{r}{r_0} e^{i(\theta - \theta_0)}\right)}. \end{aligned} \qquad (27)$$

Multipole Fields from a Current Distribution

Using the Taylor series expansion:

$$(1-\zeta)^{-1} = \sum_{n=0}^{\infty} \zeta^n, \qquad (28)$$

(valid for $|\zeta| < 1$) we can express the magnetic field (27) as:

$$B_y + iB_x = \frac{\mu_0 I_0}{2\pi r_0} e^{-i\theta_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)(\theta-\theta_0)}, \qquad (29)$$

which is valid for $r < r_0$.

$$B_y + iB_x = C_m r^{m-1} e^{i(m-1)\theta}$$

Multipole Fields from a Current Distribution

The advantage of writing the field in the form (29) is that by comparing with equation (15) we immediately see that the field is a sum over an infinite number of multipoles, with coefficients given by:

$$\frac{B_{\text{ref}}}{R_{\text{ref}}^{n-1}}(b_n + ia_n) = \frac{\mu_0 I_0}{2\pi r_0}\frac{e^{-in\theta_0}}{r_0^{n-1}}. \qquad (30)$$

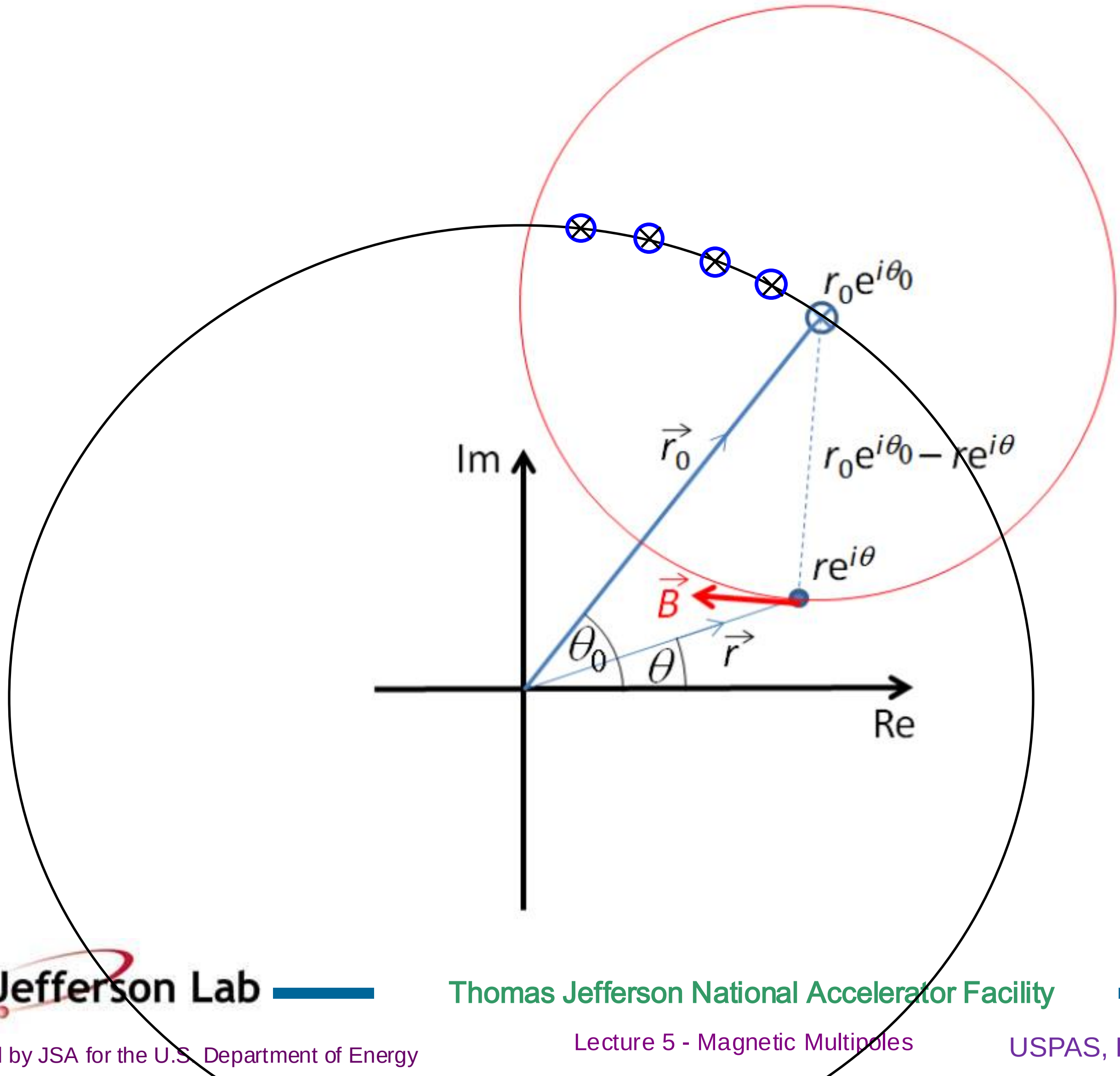
If we choose:

$$B_{\text{ref}} = \frac{\mu_0 I_0}{2\pi r_0}, \quad \text{and} \quad R_{\text{ref}} = r_0, \qquad (31)$$

we see that:

$$b_n + ia_n = e^{-in\theta_0}. \qquad (32)$$

Multipole Fields from a Current Distribution



Multipole Fields from a Current Distribution

Now, let us consider the total field generated by a set of wires distributed around a cylinder of radius r_0, such that the current flowing in a region at angle θ_0 and subtending angle $d\theta_0$ at the origin is:

$$I_0 = I_m \cos m(\theta_0 - \theta_m)\, d\theta_0, \tag{33}$$

where m is an integer.

The total field is found by integrating over all θ_0. From (29):

$$\begin{aligned} B_y + iB_x &= \frac{\mu_0 I_m}{2\pi r_0} \sum_{n=1}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\theta} \int_0^{2\pi} e^{-in\theta_0} \cos m(\theta_0 - \theta_m)\, d\theta_0 \\ &= \frac{\mu_0 I_m}{2\pi r_0} \left(\frac{r}{r_0}\right)^{m-1} e^{i(m-1)\theta}\, \pi e^{-im\theta_m}. \end{aligned}$$

We see that the cosine current distribution (33) generates a pure $2m$-pole field within the cylinder on which the current flows.

Multipole Fields from a Current Distribution

Choosing the reference field and radius (31) as we did above:

$$B_{\text{ref}} = \frac{\mu_0 I_m}{2\pi r_0}, \quad \text{and} \quad R_{\text{ref}} = r_0,$$

we find that the multipole coefficients for the field generated by the cosine current distribution (33) are:

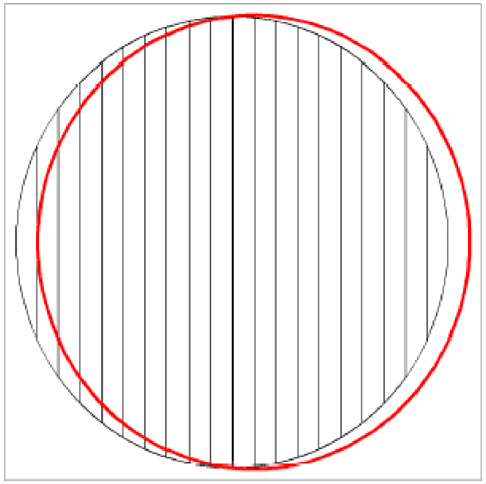
$$b_m + i a_m = \pi e^{-im\theta_m}. \tag{35}$$

For $\theta_m = 0$ or $\theta_m = \pi$, we have a normal $2m$-pole field.

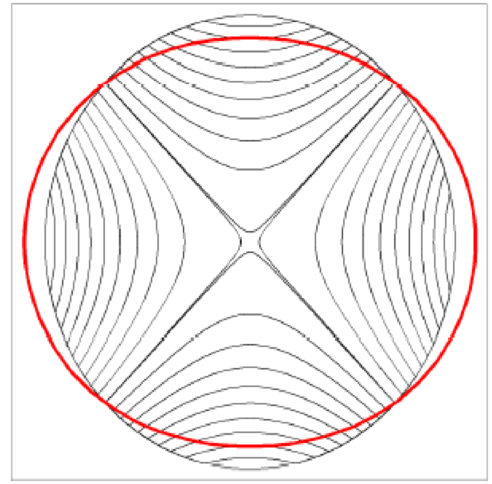
For $\theta_m = \pm\pi/2$, we have a skew $2m$-pole field.

Multipole Fields from a Current Distribution

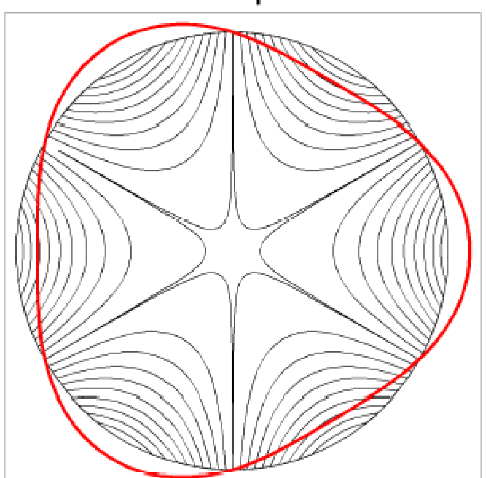
Dipole



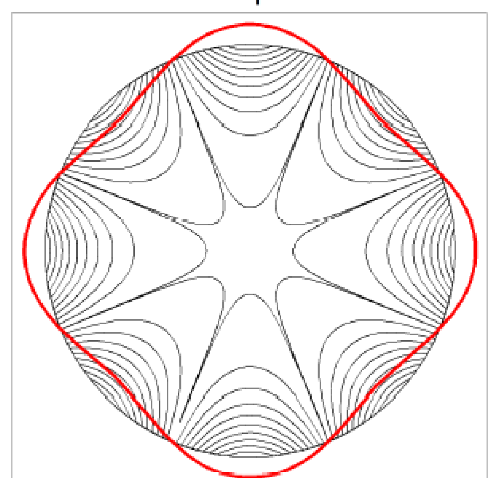
Quadrupole



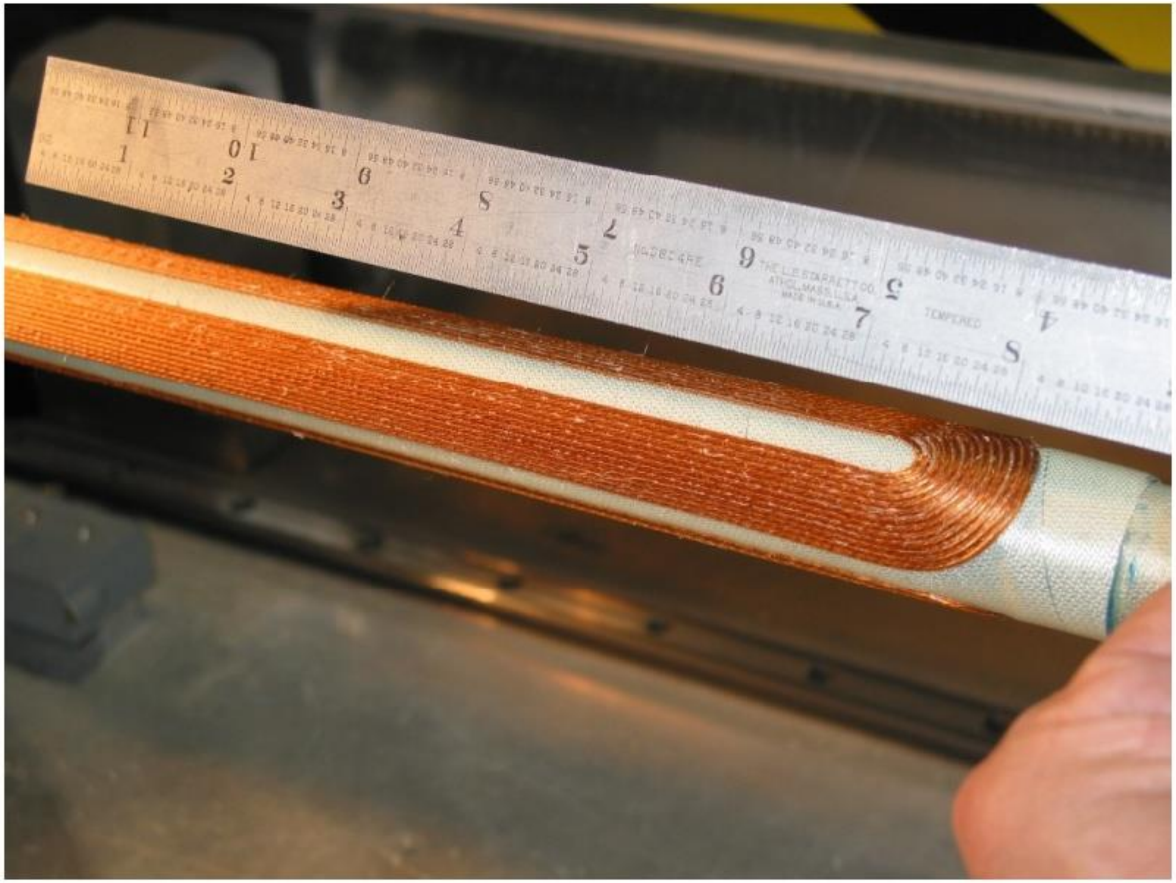
Sextupole



Octupole



Superconducting Quadrupole - Collider Final Focus



Second layer of a six-layer superconducting quadrupole developed by Brookhaven National Laboratory for a linear collider. The design goal is a gradient of 140 T/m.

Multipole Fields in an Iron-core Magnet

To generate magnetic fields of the strengths often required in accelerators using only a current distribution, the size of the current needs to be large. Usually, this means using superconductors to carry the current.

Magnetic fields of reasonable strength can also be generated using resistive conductors to drive magnetic flux in high-permeability materials.

We shall finish this lecture with a discussion of the required geometry for an iron-cored magnet to generate a pure $2m$-pole field, and the relationship between current and field strength. To keep things simple, we assume that the magnet is infinitely long in the z direction, and that the core has infinite permeability.

Multipole Fields in an Iron-core Magnet

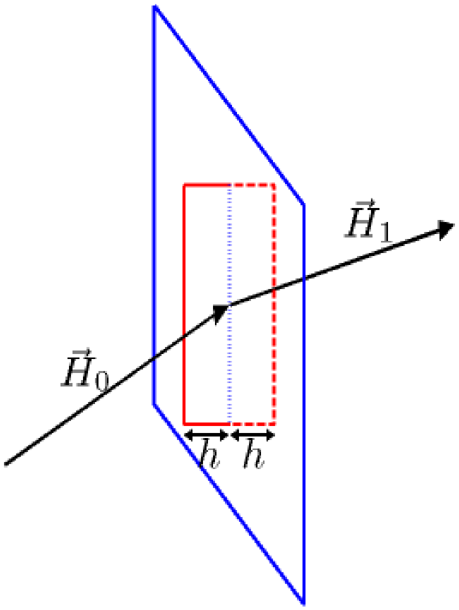
First of all, we note that the magnetic flux lines in free space must meet a material with infinite permeability normal to the surface. This we shall now show.

Consider a thin rectangular loop spanning the surface of the material. If we integrate Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (36)$$

across the surface bounded by the loop, and apply Stokes' theorem, we obtain:

$$\int_S \nabla \times \vec{H} \cdot d\vec{S} = \oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}. \qquad (37)$$



Multipole Fields in an Iron-core Magnet

Now, if we take the limit where the width of the loop tends to zero, then assuming there is no surface current, and that the time derivative of the electric displacement remain finite, we obtain:

$$H_{0t} - H_{1t} = 0, \tag{38}$$

where H_{0t} is the tangential component of the magnetic field just outside the boundary to the material, and H_{1t} is the tangential component of the magnetic field just inside the boundary.

We see that the tangential component of the magnetic field H is continuous across the boundary. Writing $B = \mu H$:

$$\frac{B_{0t}}{\mu_0} = \frac{B_{1t}}{\mu_1}. \tag{39}$$

Multipole Fields in an Iron-core Magnet

For a material with infinite permeability, assuming that the magnetic field B remains finite within the material, we see that:

$$B_{0t} = 0. \tag{40}$$

Thus, the tangential component of the field at the surface of the material vanishes; in other words, the magnetic field at the surface must be normal to the surface.

If we can shape a material (with infinite permeability) such that its surface is everywhere normal to a given $2m$-pole field, then the only field that can exist around the material will be the $2m$-pole field.

Multipole Fields in an Iron-core Magnet

To derive an explicit expression for the shape of the magnetic material in a pure $2m$-pole magnetic field, it is helpful to introduce the magnetic scalar potential, Φ. This is defined so that:

$$\vec{B} = -\nabla\Phi.$$

For static fields in free space, Maxwell's equation:

$$\nabla \times \vec{B} = 0 \qquad\qquad \nabla \times \nabla \times \vec{a} \equiv \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

is satisfied for *any* scalar field Φ; and the other Maxwell equation:

$$\nabla \cdot \vec{B} = 0$$

gives, in terms of the potential, Laplace's equation:

$$\nabla^2\Phi = 0.$$

Pierre-Simon, Marquis de Laplace (1749–1827)

Multipole Fields in an Iron-core Magnet

Since the vector $\nabla\Phi$ is always normal to a surface of constant Φ, the surface of the magnetic material of infinite permeability is always a surface of constant magnetic scalar potential.

To find the geometry for the magnetic material in a pure $2m$-pole field, we simply have to determine the appropriate magnetic scalar potential Φ, and then the equation

$$\Phi = \text{constant}$$

determines the geometry.

Multipole Fields in an Iron-core Magnet

Let us hazard a guess at the potential:

$$\Phi = -|C_m|\frac{r^m}{m}\sin(m\theta - \varphi_m).$$

Taking the gradient in cylindrical polar coordinates:

$$\begin{aligned} -\nabla\Phi &= \hat{r}\frac{\partial\Phi}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial\Phi}{\partial\theta} \\ &= \hat{r}\,|C_m|r^{m-1}\sin(m\theta - \varphi_m) + \hat{\theta}\,|C_m|r^{m-1}\cos(m\theta - \varphi_m). \end{aligned}$$

Using:

$$\hat{r} = \hat{x}\cos\theta + \hat{y}\sin\theta, \quad \text{and} \quad \hat{\theta} = -\hat{x}\sin\theta + \hat{y}\cos\theta,$$

we find:

$$-\nabla\Phi = \hat{x}\,|C_m|r^{m-1}\sin[(m-1)\theta - \varphi_m] + \hat{y}\,|C_m|r^{m-1}\cos[(m-1)\theta - \varphi_m].$$

Multipole Fields in an Iron-core Magnet

For a pure $2m$-pole field:

$$B_y + iB_x = C_m r^{m-1} e^{i(m-1)\theta}, \tag{49}$$

so:

$$B_x = |C_m| r^{m-1} \sin[(m-1)\theta - \varphi_m] , \tag{50}$$

$$B_y = |C_m| r^{m-1} \cos[(m-1)\theta - \varphi_m] . \tag{51}$$

Comparing with equation (48), we conclude that the scalar potential (45):

$$\boxed{\Phi = -|C_m| \frac{r^m}{m} \sin(m\theta - \varphi_m)} \tag{52}$$

generates the pure $2m$-pole field:

$$\boxed{B_y + iB_x = -\nabla\Phi = C_m (x + iy)^{m-1}.} \tag{53}$$

Multipole Fields in an Iron-core Magnet

Since the surface of the magnetic material must be surface of constant potential (assuming infinite permeability), we see that the surface of the material in a pure $2m$-pole field must be given by:

$$r^m \sin(m\theta - \varphi_m) = \text{constant}, \tag{54}$$

or:

$$r = \sqrt[m]{\frac{\text{constant}}{\sin(m\theta - \varphi_m)}}. \tag{55}$$

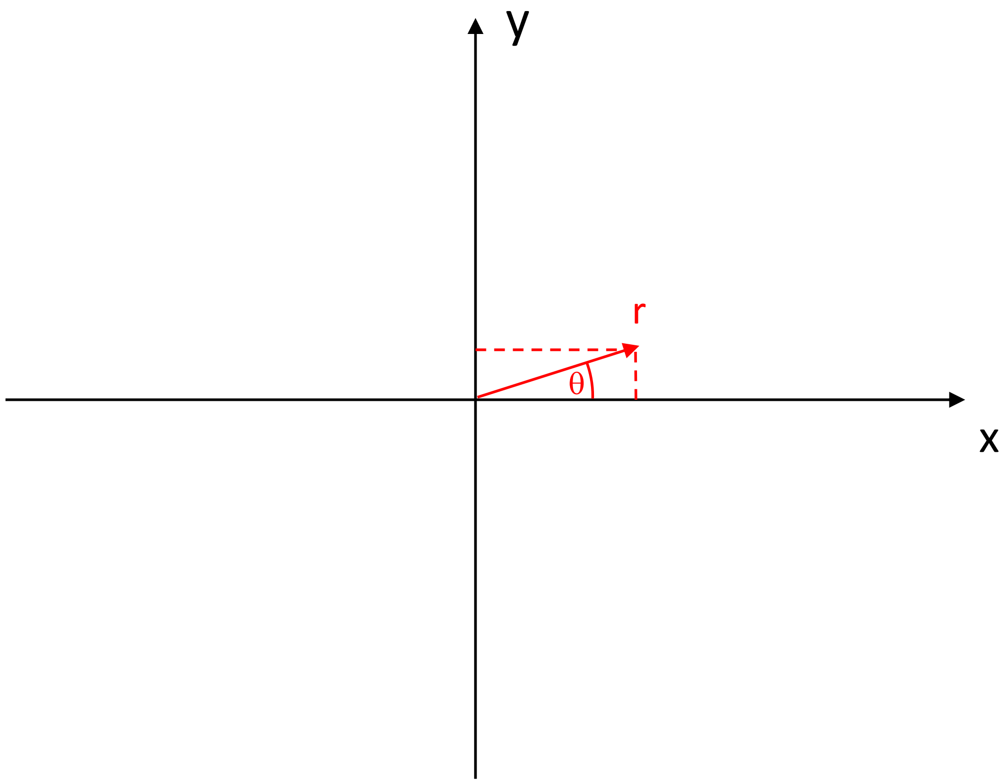
φ_m is the phase angle of C_m^*. If $\varphi_m = 0$, then C_m is real, and we generate a normal $2m$-pole field. If $\varphi_m = \pi/2$, then C_m is imaginary, and we generate a skew $2m$-pole field.

Pole Shape – Quadrupole

m = 2

$$r^m \sin(m\theta - \varphi_m) = C$$

$\varphi_m = 0$

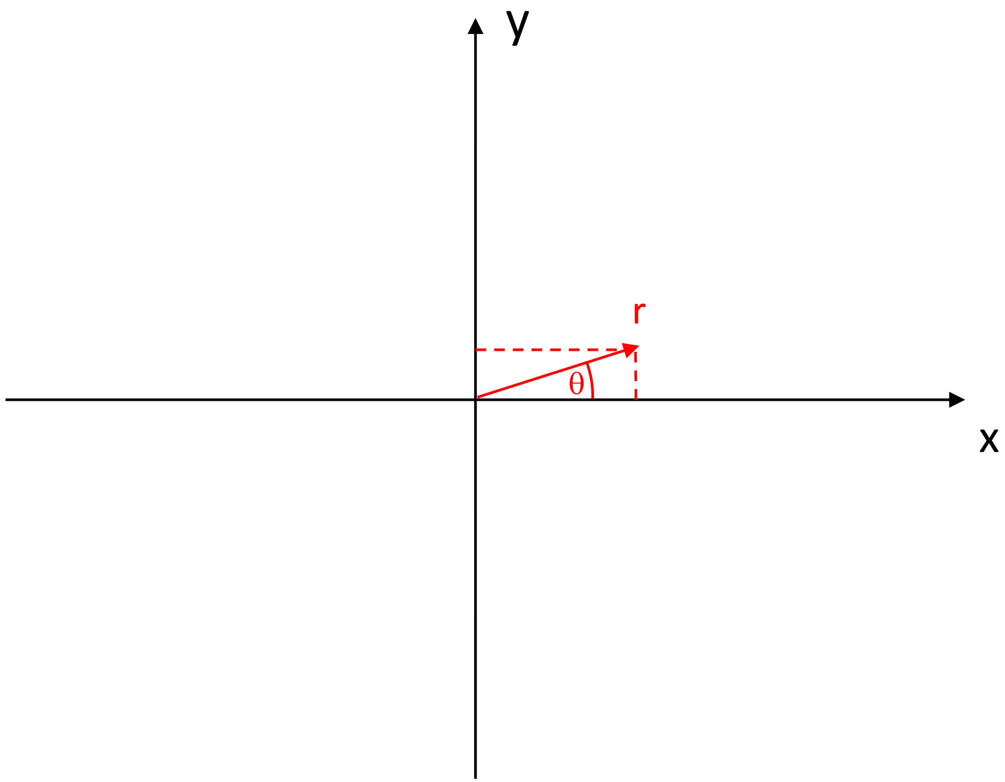


Pole Shape – Dipole

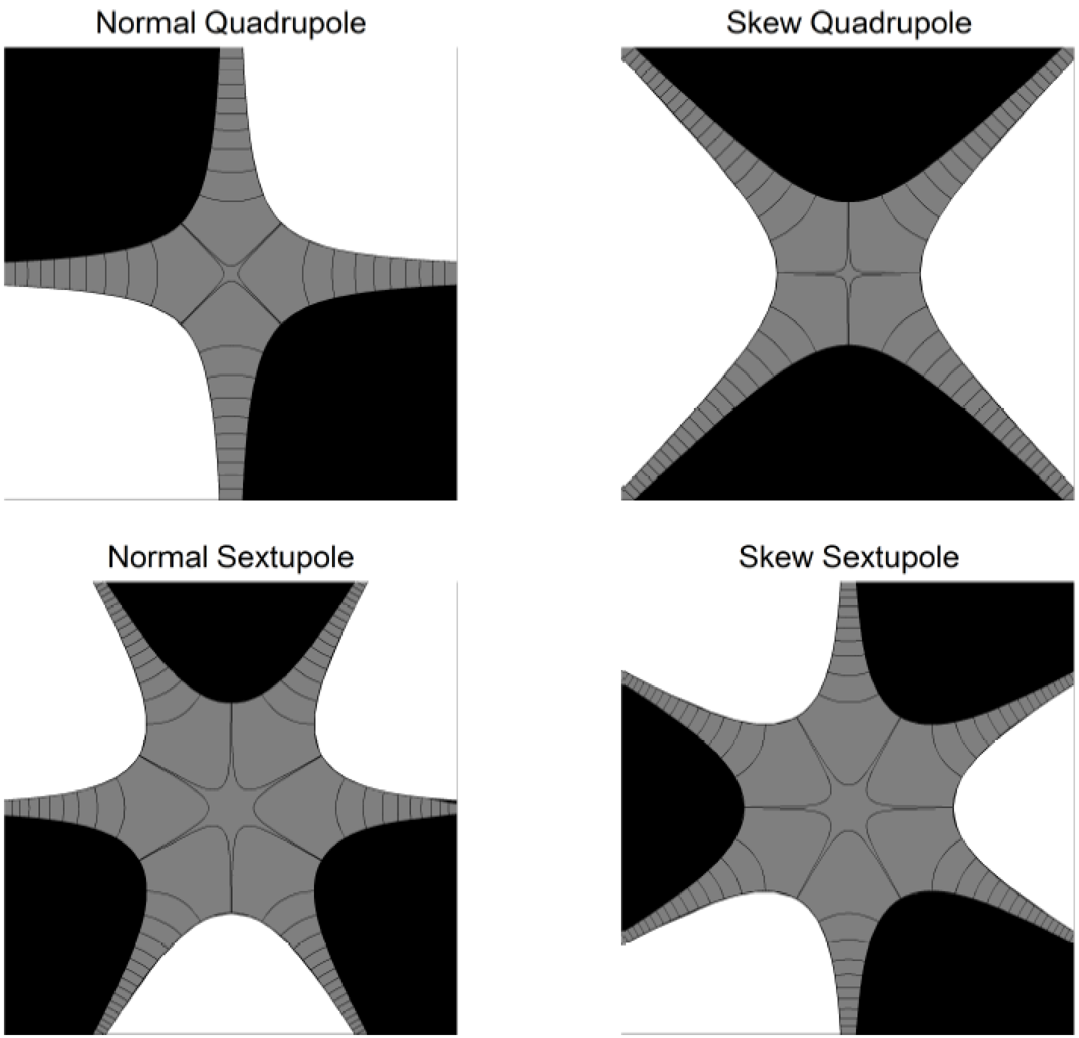
m = 1

$$r^m \sin(m\theta - \varphi_m) = C$$

$\varphi_m = 0$



Multipole Fields in an Iron-core Magnet



Multipole Fields in an Iron-core Magnet

Our final task is to calculate the field strength in an iron-cored magnet for a given number of ampere-turns around each pole. To do this, we can consider just a normal $2m$-pole, since skew $2m$-poles are simply rotations of normal $2m$-poles.

We assume that the magnetic field is generated by wires carrying currents between the poles, with the wires parallel to the z axis, and positioned a large distance from the axis. Since the distance from the centre of the magnet to the currents is large, we can neglect the field arising "directly" from the current, and consider only the field arising from magnetisation of the iron.

Furthermore, we maintain symmetry by placing equal currents between each pair of poles, alternating in direction from one set of wires to the next.

Generating Multipole Fields in an Iron-core Magnet

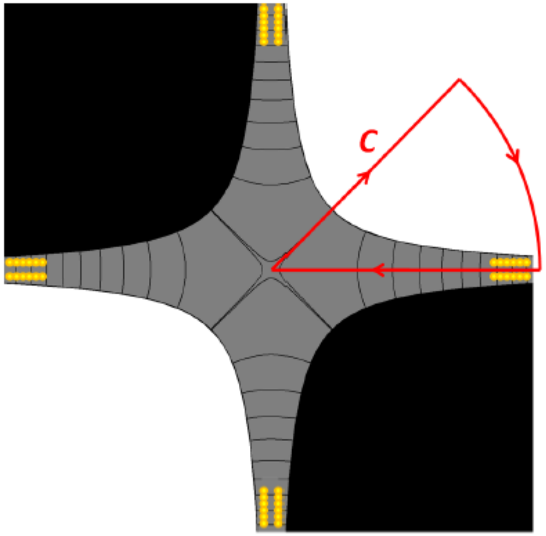
We again use Maxwell's equation:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \qquad (56)$$

Now we integrate across a surface in the x-y plane, bounded by the contour C defined by the lines:

$$\theta = 0, \quad \text{and} \quad \theta = \frac{\pi}{2m}, \qquad (57)$$

and closed at $r \to \infty$.



Again applying Stokes' theorem, we obtain:

$$\oint_C \vec{H} \cdot d\vec{\ell} = NI, \qquad (58)$$

where there are $2N$ wires carrying current I between each pair of poles.

Note that conventionally, the current is supplied by a coil of N turns around each pole; thus the total number of wires between each adjacent pair of poles is $2N$.

Generating Multipole Fields in an Iron-core Magnet

We can break the path integral into two segments: C_0 in vacuum with permeability μ_0, and C_1 inside the magnetic material with permeability μ:

$$\int_{C_0} \frac{\vec{B}}{\mu_0} \cdot d\vec{\ell} + \int_{C_1} \frac{\vec{B}}{\mu} \cdot d\vec{\ell} = NI. \tag{59}$$

In the limit $\mu \to \infty$, the segment of the integral inside the magnetic material vanishes. Furthermore along the segment $\theta = 0$, the field is perpendicular to the path, so makes no contribution to the path integral.

We are left with:

$$\int_0^{r_0} B_r \, dr = \mu_0 NI, \tag{60}$$

where r_0 is the radius of the largest circle that can be inscribed within the pole tips of the magnet.

Generating Multipole Fields in an Iron-core Magnet

The radial field component along $\theta = \pi/2m$ is given by:

$$B_r = B_{\text{ref}} b_m \left(\frac{r}{R_{\text{ref}}}\right)^{m-1} . \qquad (61)$$

Let us choose $R_{\text{ref}} = r_0$, and $B_{\text{ref}} = B_r(r_0, \pi/2m) = B_0$. Then, $b_m = 1$:

$$B_r = B_0 \left(\frac{r}{r_0}\right)^{m-1} , \qquad (62)$$

and we obtain:

$$\int_0^{r_0} B_r \, dr = \frac{B_0 r_0}{m} = \mu_0 N I. \qquad (63)$$

Generating Multipole Fields in an Iron-core Magnet

Therefore, the field is given by:

$$B_y + iB_x = \frac{m\mu_0 NI}{r_0}\left(\frac{x+iy}{r_0}\right)^{m-1}. \tag{64}$$

The multipole gradient is given by:

$$\frac{\partial^{m-1} B_y}{\partial x^{m-1}} = \frac{m!\mu_0 NI}{r_0^m}. \tag{65}$$

For example, for a quadrupole magnet ($m = 2$), the gradient is given by:

$$\frac{\partial B_y}{\partial x} = \frac{2\mu_0 NI}{r_0^2}. \tag{66}$$

Re-cap on Magnetic Multipoles

We have shown that:

- multipole fields satisfy Maxwell's equations in free space;
- a pure $2m$-pole field can be generated by a $\cos(m\theta)$ current distribution on the surface of a cylinder;
- a pure $2m$-pole field can be generated by an iron-cored magnet, whose pole tips follow surfaces of constant magnetic scalar potential.

Of course, the expressions we have derived here are only exactly correct with ideal (and rather impractical) conditions on the geometry and material properties.

Re-cap on Multipole Fields

$$B_y + iB_x = C_n(x+iy)^{n-1} = C_n r^{n-1} e^{i(n-1)\theta} \tag{1}$$

with $B_z =$ constant, provided valid solutions to Maxwell's equations in free space.

We also saw that such a field could be generated by a current flowing parallel to the z-axis, on a cylinder of radius r_0, with distribution:

$$I(\theta) = I_n \cos n(\theta - \theta_n). \tag{2}$$

In this case, the field is given by:

$$B_y + iB_x = \frac{\mu_0 I_n}{2r_0} e^{-in\theta_n} \left(\frac{r}{r_0}\right)^{n-1} e^{i(n-1)\theta}. \tag{3}$$

Re-cap on Multipole Fields

Multipole fields can also be generated by currents flowing in wires wound around iron poles. For a pure $2n$-pole field, the shape of the surface of the iron pole must match a surface of constant scalar potential, Φ, given by:

$$\Phi = -|C_n|\frac{r^n}{n}\sin n(\theta - \theta_n). \tag{4}$$

The field is given by:

$$\vec{B} = -\nabla\Phi. \tag{5}$$

If each pole in an "ideal" $2n$-pole magnet is wound with N turns of wire carrying current I, then the multipole gradient of the field is:

$$\frac{\partial^{n-1}B_y}{\partial x^{n-1}} = \frac{n!\mu_0 NI}{r_0^n}, \tag{6}$$

where r_0 is the radius of the largest cylinder that can be inscribed between the poles.

Magnetic Multipoles – Application to Magnets

- Deduce that the symmetry of a magnet imposes constraints on the possible multipole field components, even if we relax the constraints on the material properties and other geometrical properties;
- Consider different techniques for deriving the multipole field components from measurements of the fields within a magnet;
- Discuss the solutions to Maxwell's equations that may be used for describing fields in three dimensions.

Allowed and Forbidden Harmonics

In general, from equation (1), a pure $2n$-pole field can be written:

$$B_y + iB_x = |C_n| e^{-in\theta_n} r^{n-1} e^{i(n-1)\theta}, \tag{7}$$

where r and θ are the polar coordinates within the magnet, and θ_n is the angle by which the magnet is rolled around the z axis.

We cannot design a realistic magnet to produce a pure $2n$-pole field. The materials will have finite permeability and finite dimensions, and may saturate.

More generally, a field consists of a superposition of $2n$-pole fields:

$$B_y + iB_x = \sum_{n=1}^{\infty} |C_n| e^{-in\theta_n} r^{n-1} e^{i(n-1)\theta}. \tag{8}$$

Allowed and Forbidden Harmonics

However, in the design of a $2n$-pole magnet, we *can* impose a perfect symmetry under rotations through $2\pi/n$ about the z axis.

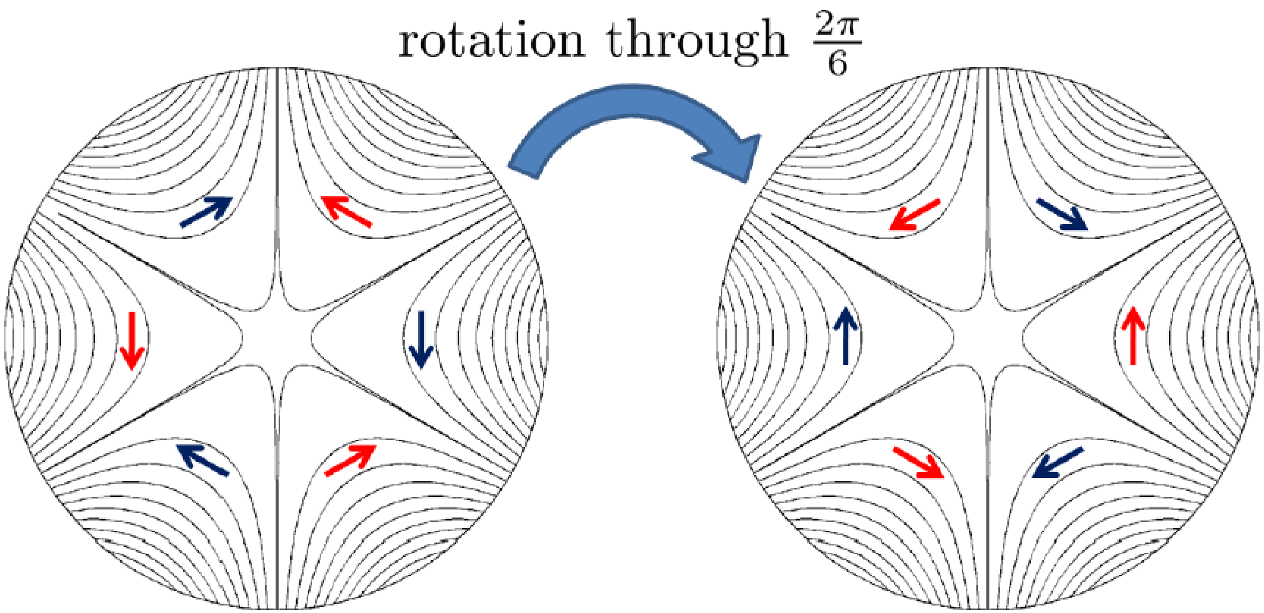
In fact, we see from equation (7):

$$B_y + iB_x = |C_n|e^{-in\theta_n}r^{n-1}e^{i(n-1)\theta}, \tag{9}$$

that under a rotation by π/n, i.e. $\theta_n \mapsto \theta_n + \pi/n$, the field changes sign:

$$\vec{B} \mapsto -\vec{B}. \tag{10}$$

Allowed and Forbidden Harmonics



Since this symmetry is imposed by the geometry of the magnet, any multipole field within the magnet must obey this symmetry. This restricts the multipole components that may be present, at least in the design.

Allowed and Forbidden Harmonics

Consider a field given by:

$$B_y + iB_x = |C_n|e^{-in\theta_n}r^{n-1}e^{i(n-1)\theta} + |C_m|e^{-im\theta_m}r^{m-1}e^{i(m-1)\theta}. \tag{11}$$

If the geometry of the magnet is such that the field simply changes sign under a rotation through π/n, then the "extra" harmonic must satisfy:

$$e^{-i\pi\frac{m}{n}} = -1, \tag{12}$$

therefore:

$$\frac{m}{n} = 1,\ 3,\ 5,\ 7\ \ldots \tag{13}$$

We see that only *higher* harmonics are allowed; and the indices of the higher harmonics must be an odd integer multiple of the main harmonic.

Allowed and Forbidden Harmonics

For a dipole, $n = 1$, so the allowed harmonics are $m = 3, 5, 7 \ldots$
For a quadrupole, $n = 2$, and the allowed harmonics are $m = 6, 10, 14 \ldots$

Of course, this argument cannot tell us the strengths of the allowed harmonics: those depend on the details of the design.

It is also a fact that a magnet, when fabricated, will never exhibit perfectly the symmetry with which it was designed. Therefore, a real physical magnet will generally include *all* harmonics to some extent, not just the harmonics allowed by the ideal symmetry.

However, for a carefully fabricated magnet, the harmonics forbidden by the ideal symmetry should be small in comparison to the allowed harmonics; and it should be possible to predict the sizes of the allowed harmonics accurately from the design.

Measuring Multipoles

Knowing the multipole components of magnets in an accelerator is important for understanding the beam dynamics.

Construction tolerances will mean that the strengths of the multipoles present in the magnet will differ from those in the design.

This leads us to consider how to determine the multipole components from measurements of the magnetic field.

There are many possible approaches to the problem: we shall consider two, for illustration, and only go as far as necessary to understand some of the pros and cons.

Measuring Multipoles in Cartesian Basis

The field may be represented as:

$$B_y + iB_x = \sum_n C_n (x + iy)^{n-1}, \qquad (14)$$

where the real parts of the coefficients C_n give the normal multipole strengths, and the imaginary parts give the skew multipole strengths.

If we take a set of measurements of B_y and B_x along the x axis, then $y = 0$ at each measurement point, and the normal multipoles can be found by fitting a polynomial to B_y *vs* x, and the skew multipoles can be found by fitting a polynomial to B_x *vs* x.

Measuring Multipoles in Cartesian Basis

The real problem is that mathematically, the basis functions that we use to fit the data (monomials in x and, possibly, y) are not orthogonal. This means that data constructed from one monomial can be fitted, with non-zero strength, with a completely different monomial.

Although this does not invalidate the technique altogether, it does make it a little difficult to apply accurately. Ideally, we need to know in advance which multipole components are present.

However, there is a more robust technique...

Measuring Multipoles in Polar Basis

Instead of expressing the field in Cartesian coordinates, let us write the same field in polar coordinates:

$$B_y + iB_x = \sum_n C_n r^{n-1} e^{i(n-1)\theta}. \tag{15}$$

Now suppose that we take a set of measurements of B_y and B_x at fixed $r = r_0$, but at M equally spaced steps in $\theta = 2\pi m/M$, where $m = 1 \ldots M$.

We then notice that:

$$\begin{aligned} \sum_{m=1}^{M} (B_y + iB_x)_m e^{-2\pi i(n'-1)\frac{m}{M}} &= \sum_{m=1}^{M} \sum_n C_n r_0^{n-1} e^{2\pi i(n-n')\frac{m}{M}} \\ &= M C_{n'} r_0^{n'-1}. \end{aligned} \tag{16}$$

Hence:

$$C_n = \frac{1}{M r_0^{n-1}} \sum_{m=1}^{M} (B_y + iB_x)_m e^{-2\pi i(n-1)\frac{m}{M}}. \tag{17}$$

Measuring Multipoles in Polar Basis

The advantage of using polar coordinates, over Cartesian coordinates, is that the basis functions, $e^{i(n-1)\theta}$, are orthogonal. Mathematically, we have:

$$\sum_{m=1}^{M} e^{2\pi i(n-1)\frac{m}{M}} e^{-2\pi i(n'-1)\frac{m}{M}} = \begin{cases} 0 & \text{if} \quad n \neq n' \\ M & \text{if} \quad n = n' \end{cases} \tag{18}$$

The orthogonality means that the value we determine for one multipole component is completely unaffected by the presence of other multipole components.

Determining the multipole coefficients C_n amounts to carrying out a discrete Fourier transform on the field data measured on a cylinder inscribed through the magnet.

Measuring Multipoles in Polar Basis

A further advantage of the polar basis comes from the fact that the radius of the cylinder on which the field data are collected appears as $1/r_0^{n-1}$ in the expression for the coefficients C_n, equation (17).

Suppose that there is some error in the field measurements. This will lead to some error in the values of C_n that we determine. If we reconstruct the field (e.g. for particle tracking), then there will be some error in the calculated field. However, this error will decrease as r^{n-1}, as we go towards the centre of the magnet (where the beam is).

Of course, if we try to extrapolate the field outside the cylinder of radius r_0, then any errors will *increase* as some power of the distance from the centre.

Advantages of Mode Decompositions

There are some important advantages to describing a field in terms of a mode decomposition, instead of a set of numerical field values on a grid:

- A description of the field in terms of mode coefficients is very much more compact than a description in terms of numerical field data.
- A field constructed from mode coefficients is guaranteed to satisfy Maxwell's equations: numerical field data are not.
- Measurement noise can be "smoothed" by suppressing higher-order modes.
- Errors can be represented in a realistic way by introducing higher-order modes.
- A number of beam dynamics analysis tools require mode decompositions.

Three-dimensional Fields

The polar basis for fitting multipole field components generalises nicely to three-dimensional fields. But, since we have not so far discussed such fields at all, before showing how the field fitting works, we need to discuss solutions to Maxwell's equations for three-dimensional magnets.

As before, the relevant equations are:

$$\nabla \cdot \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{B} = 0. \tag{19}$$

Any field that satisfies these equations is a possible magnetic field in free space. So far, we have considered only multipole fields, that are independent of one coordinate; but this is not very realistic.

Three-dimensional Fields

A field that satisfies Maxwell's equations (19) is given by:

$$B_x = -B_0 \frac{k_x}{k_y} \sin k_x x \, \sinh k_y y \, \sin k_z z, \tag{20}$$

$$B_y = B_0 \cos k_x x \, \cosh k_y y \, \sin k_z z, \tag{21}$$

$$B_z = B_0 \frac{k_z}{k_y} \cos k_x x \, \sinh k_y y \, \cos k_z z, \tag{22}$$

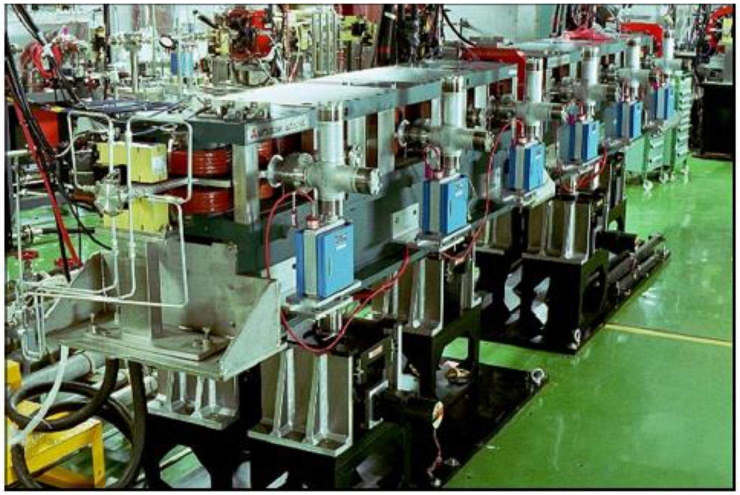
where:

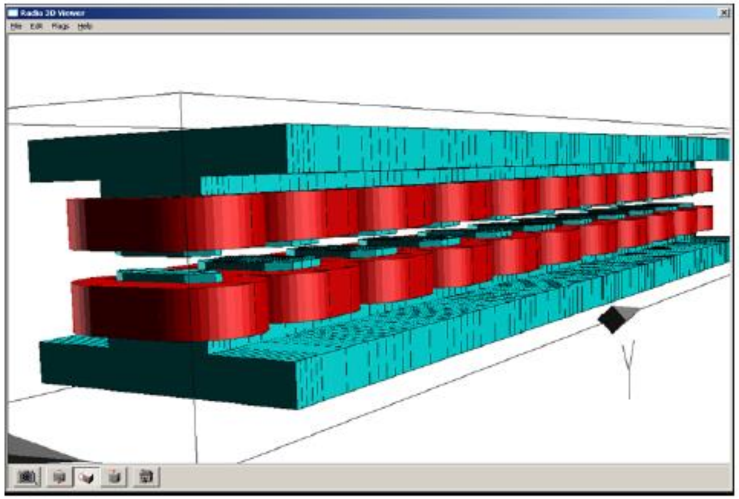
$$k_y^2 = k_x^2 + k_z^2. \tag{23}$$

There are a number of variations on this field, for example, with the hyperbolic function appearing in the x or z coordinates; or, with different phases in x and/or z. However, the above representation is particularly convenient for describing insertion devices (wigglers and undulators), as we shall now discuss.

Three-dimensional Fields

$$
\begin{aligned}
B_x &= -B_0 \frac{k_x}{k_y} \sin k_x x \, \sinh k_y y \, \sin k_z z, \\
B_y &= B_0 \cos k_x x \, \cosh k_y y \, \sin k_z z, \\
B_z &= B_0 \frac{k_z}{k_y} \cos k_x x \, \sinh k_y y \, \cos k_z z,
\end{aligned}
$$





Normal-conducting electromagnetic wiggler at the KEK Accelerator Test Facility.

Three-dimensional Fields

If we take $k_x = 0$, then the field becomes:

$$B_x = 0, \tag{24}$$

$$B_y = B_0 \cosh k_z y \sin k_z z, \tag{25}$$

$$B_z = B_0 \sinh k_z y \cos k_z z. \tag{26}$$

The above equations describe a field that varies sinusoidally in z, and has no horizontal (x) component at all. This is a field that could only occur in an insertion device with infinite length, and infinite width.

Three-dimensional Fields

A better description would take account of the fact that the longitudinal variation of the field will not be perfectly sinusoidal. We can account for this by superposing fields with different values of k_z:

$$B_x = 0, \tag{27}$$

$$B_y = \int \tilde{B}(k_z) \cosh k_z y \, \sin k_z z \, dk_z, \tag{28}$$

$$B_z = \int \tilde{B}(k_z) \sinh k_z y \, \cos k_z z \, dk_z. \tag{29}$$

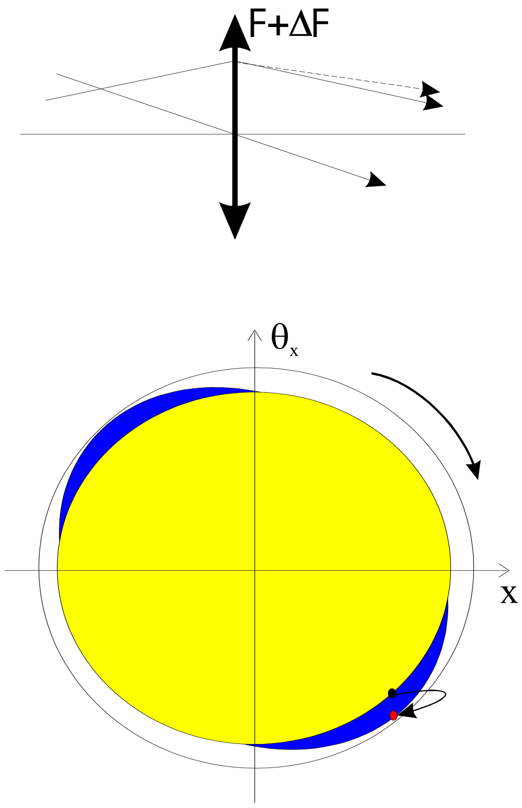
We see that if we take measurements of B_y as a function of z in the plane $y = y_0$ (for fixed y_0), then we can obtain the *mode coefficients* $\tilde{B}(k_z)$ by a (discrete) Fourier transform.

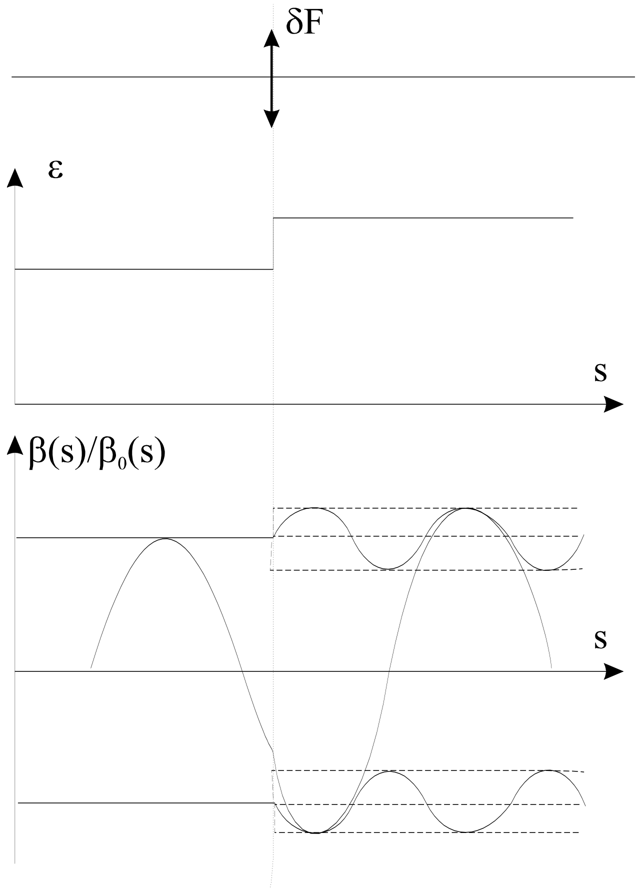
This allows us to reconstruct all field components, at all locations within the field.

Magnet Design – Summary

- Symmetries in multipole magnets restrict the multipole components that can be present in the field.
- It is useful to be able to find the multipole components in a given field from numerical field data: but this must be done carefully, if the results are to be accurate.
- Usually, it is advisable to calculate multipole components using field data on a surface enclosing the region of interest: any errors or residuals will decrease exponentially within that region, away from the boundary. Outside the boundary, residuals will increase exponentially.
- Techniques for finding multipole components in two dimensional fields can be generalized to three dimensions, allowing analysis of fringe fields and insertion devices.
- In two or three dimensions, it is possible to use a Cartesian basis for the field modes; but a polar basis is sometimes more convenient.

- Focusing 'point' error perturbs the betatron motion leading to the Courant-Snyder invariant change:





Beam envelope and beta-function oscillate at double the betatron frequency

Appendix A - Field Error Tolerances

- Single point mismatch as measured by the Courant-Snyder invariant change:

$$\varepsilon' = \beta(\theta+\delta\theta)^2 + 2\alpha(\theta+\delta\theta)x + \gamma x^2$$
$$= \varepsilon + 2\left(\beta\theta + \alpha x\right)\delta\theta + \beta\delta\theta^2 \quad ,$$

$$x = \sqrt{\varepsilon\beta}\,\sin\mu, \quad \theta = \sqrt{\frac{\varepsilon}{\beta}}\,\sin\mu\left(\cos\mu - \alpha\sin\mu\right)$$

- Each source of field error (magnet) contributes the following Courant-Snyder variation

$$\delta\varepsilon = 2\sqrt{\varepsilon\beta}\,\cos\mu\,\delta\theta + \beta\delta\theta^2 \quad ,$$

$$\delta\theta = \frac{\int B^{grad}dl}{B\rho} = \sum_{m=1}\delta\phi_m x^m \;,where \quad \delta\phi_m = \int \delta k_m dl$$

here, m =1 quadrupole, m =2 sextupole, m=3 octupole, etc

$$\delta\varepsilon = 2\sqrt{\varepsilon\beta}\,\sum_{m=1}\left(\sqrt{\varepsilon\beta}\right)^m \delta\phi_m \cos\mu\,\sin^m\mu \;+ \beta\left(\sum_{m=1}\left(\sqrt{\varepsilon\beta}\right)^m \delta\phi_m\,\sin^m\mu\right)^2,$$

Appendix A - Field Error Tolerances

- multipole expansion coefficients of the azimuthal magnetic field, B_θ - Fourier series representation in polar coordinates at a given point along the trajectory):

$$B_\theta(r,\theta) = \sum_{m=2} \left(\frac{r}{r_0}\right)^{m\text{-}1} \left(B_m \cos m\theta + A_m \sin m\theta\right)$$

- multipole gradient and integrated geometric gradient:

$$G_m = \frac{1}{r_0{}^m} \mathrm{B}_{m+1} \quad \left[kGauss\ cm^{-m}\right] \qquad k_n = \frac{G_n}{B\rho} \quad \left[cm^{-(n+1)}\right] \qquad \phi_n = \frac{\int G_n dl}{B\rho} \quad \left[cm^{-n}\right]$$

Appendix A - Field Error Tolerances

- Cumulative mismatch along the lattice (N sources):

$$\varepsilon_N = \varepsilon \prod_{n=1}^{N} \left(1 + 2\beta \sum_{m=1} \left(\sqrt{\varepsilon\beta}\right)^{m-1} \delta\phi_m \cos\mu \, \sin^m \mu \;\; + \beta^2 \left(\sum_{m=1} \left(\sqrt{\varepsilon\beta}\right)^{m-1} \delta\phi_m \, \sin^m \mu \right)^2 \right) \quad ,$$

- Standard deviation of the Courant-Snyder invariant is given by:

$$\frac{\sigma_\varepsilon}{\varepsilon} = \frac{\sqrt{\left\langle \delta\varepsilon^2 \right\rangle - \left\langle \delta\varepsilon^2 \right\rangle^2}}{\varepsilon} = \sqrt{\sum_{i=1}^{N} \left[2\beta_i \sum_{m=1} \left(\sqrt{\varepsilon\beta_i}\right)^{m-1} \delta\phi_m \left\langle \cos\mu \, \sin^m \mu \right\rangle \;\; + \beta_i^2 \left\langle \left(\sum_{m=1} \left(\sqrt{\varepsilon\beta_i}\right)^{m-1} \delta\phi_m \, \sin^m \mu \right)^2 \right\rangle \right]}$$

- Assuming weakly focusing lattice (uniform beta modulation) the following averaging (over the betatron phase) can by applied:

$$\langle \ldots \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\mu \ldots$$

Appendix A - Field Error Tolerances

- Some useful integrals …. :

$$\left\langle \cos\mu\ \sin^m\mu \right\rangle = 0 \quad ,$$

$$\left\langle \sin^m\mu \right\rangle = \frac{m-1}{m}\left\langle \sin^{m-2}\mu \right\rangle = \begin{cases} 0 & \text{m odd} \\ \dfrac{(m-1)!!}{m!!} & \text{m even} \end{cases}$$

will reduce the coherent contribution to the C-S variance as follows:

$$\frac{\sigma_\varepsilon}{\varepsilon} = \sqrt{\sum_{i=1}^{N}\left[2\beta_i \sum_{m=1}\left(\sqrt{\varepsilon\beta_i}\right)^{m-1}\delta\phi_m \cancel{\left\langle \cos\mu\ \sin^m\mu \right\rangle} + \beta_i^2 \left\langle \left(\sum_{m=1}\left(\sqrt{\varepsilon\beta_i}\right)^{m-1}\delta\phi_m\ \sin^m\mu\right)^2 \right\rangle\right]}$$

- Including the first five multipoles yields:

$$\frac{\sigma_\varepsilon}{\varepsilon} = \sqrt{\sum_{i=1}^{N}\left\{\beta_i^2\left[\delta\phi_1^2\underbrace{\left\langle \sin^2\mu \right\rangle}_{\frac{1}{2}} + \varepsilon\beta_i\left(\delta\phi_2^2 + 2\delta\phi_1\delta\phi_3\right)\underbrace{\left\langle \sin^4\mu \right\rangle}_{\frac{1}{2}\,\frac{3}{4}} + \left(\varepsilon\beta_i\right)^2\left(\delta\phi_3^2 + 2\delta\phi_1\delta\phi_5 + 2\delta\phi_2\delta\phi_4\right)\underbrace{\left\langle \sin^6\mu \right\rangle}_{\frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}} + \ldots\right]\right\}}$$

Appendix A - Field Error Tolerances

- Beam radius at a given magnet is : $a_i = \frac{1}{2}\sqrt{\varepsilon\beta_i}$

- One can define a 'good fileld radius' for a given type of magnet as: $a = Max(a_i)$

- Assuming the same multipole content for all magnets in the class one gets:

$$\boxed{\frac{\sigma_\varepsilon}{\varepsilon} = \sqrt{\sum_{i=1}^{N}\frac{1}{2}\beta_i^{\,2}} \times \sqrt{\delta\phi_1^{\,2} + \frac{3}{2}a^2\left(\delta\phi_2^{\,2} + 2\delta\phi_1\delta\phi_3\right) + \frac{5}{2}a^4\left(\delta\phi_3^{\,2} + 2\delta\phi_1\delta\phi_5 + 2\delta\phi_2\delta\phi_4\right) + \ldots}}$$

- The first factor purely depends on the beamline optics (focusing), while the second one describes field tolerance (nonlinearities) of the magnets:

$$\Delta\Phi = \sqrt{\delta\phi_1^{\,2} + \frac{3}{2}a^2\left(\delta\phi_2^{\,2} + 2\delta\phi_1\delta\phi_3\right) + \frac{5}{2}a^4\left(\delta\phi_3^{\,2} + 2\delta\phi_1\delta\phi_5 + 2\delta\phi_2\delta\phi_4\right) + \ldots}$$

Appendix B - The vector potential

- A scalar potential description of the magnetic field has been very useful to derive the shape for the pole face of a multipole magnet.

The scalar potential Φ is defined such that:

$$\vec{B} = -\nabla\Phi. \tag{49}$$

With this definition, the equation $\nabla \times \vec{B} = 0$ is automatically satisfied. The equation $\nabla \cdot \vec{B} = 0$ leads to Laplace's equation for the scalar potential:

$$\nabla^2\Phi = 0. \tag{50}$$

Appendix B - The Vector Potential

However, the scalar potential is only defined in the absence of currents. More generally, we need to use a vector potential $\vec{A}$. In fact, in the most general case of time-dependent electric and magnetic fields, we need both a scalar potential ϕ, and a vector potential $\vec{A}$:

$$\vec{B} = \nabla \times \vec{A}, \quad \text{and} \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}. \tag{51}$$

Some important methods for beam dynamics analysis use the potentials ϕ and $\vec{A}$, rather than the fields. It is therefore useful to have expressions for the potentials corresponding to the expressions for the fields we have derived in the main part of these lectures.

Appendix B - The vector potential

For the case of interest here (a magnetostatic field, and zero electric field), we can assume that ϕ is independent of position, and $\vec{A}$ is independent of time.

If we allow the presence of nonmagnetic materials ($\mu = \mu_0$) carrying an electric current density $\vec{J}$, then substituting from (51) into Maxwell's equations gives:

$$\nabla \cdot \vec{B} = \nabla \cdot \nabla \times \vec{A} \equiv 0, \tag{52}$$

and:

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A} \equiv \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}. \tag{53}$$

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Maxwell's equation $\nabla \cdot \vec{B} = 0$ is automatically satisfied by any vector potential $\vec{A}$, by virtue of a vector identity (the divergence of the curl of any vector field is zero).

Maxwell's equation $\nabla \times \vec{H} = \vec{J}$ (assuming static fields) is satisfied, if the vector potential $\vec{A}$ satisfies:

$$\nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu_0 \vec{J}. \tag{54}$$

Now, we observe that since $\vec{B} = \nabla \times \vec{A}$, and $\nabla \times \nabla\psi$ is identically zero for any scalar field ψ, we can define a new potential $\vec{A}' = \vec{A} + \nabla\psi$ that gives exactly the same field as $\vec{A}$. We can use this property of the fields and potentials, known as *gauge invariance*, to simplify equation (54).

Appendix B - The vector potential

Suppose that we have a vector potential $\vec{A}$ for which:

$$\nabla \cdot \vec{A} = f, \tag{55}$$

where f is some function of position. Then, if we define:

$$\vec{A}' = \vec{A} + \nabla\psi, \tag{56}$$

where ψ satisfies Poisson's equation:

$$\nabla^2\psi = -f, \tag{57}$$

then $\vec{A}'$ gives the same field $\vec{B}$ as $\vec{A}$, and:

$$\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2\psi = 0. \tag{58}$$

In other words, if we can solve Poisson's equation (57) for ψ, then we can make a *gauge transformation* to a vector potential that has vanishing divergence.

Appendix B - The vector potential

Let us suppose that we find a vector potential that has vanishing divergence:

$$\nabla \cdot \vec{A} = 0. \tag{59}$$

Equation (59) amounts to a condition that specifies a particular choice of gauge: this particular choice (i.e. with zero divergence) is known as the Coulomb gauge. It is useful, because equation (54) for the vector potential then takes the simpler form:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}. \tag{60}$$

This is Poisson's equation, which has the standard solution:

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3 r'. \tag{61}$$

Appendix B - The vector potential

Now, consider the potential given by:

$$A_x = 0, \quad A_y = 0, \quad A_z = -\mathrm{Re}\frac{C_n(x+iy)^n}{n}. \tag{62}$$

Taking derivatives, we find that:

$$\frac{\partial A_z}{\partial x} = -\mathrm{Re}\, C_n(x+iy)^{n-1}, \quad \text{and} \quad \frac{\partial A_z}{\partial y} = \mathrm{Im}\, C_n(x+iy)^{n-1}. \tag{63}$$

Hence:

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A}, \\ &= \left(\frac{\partial A_z}{\partial y}, -\frac{\partial A_z}{\partial x}, 0\right), \\ &= \left(\mathrm{Im}\, C_n(x+iy)^{n-1}, \mathrm{Re}\, C_n(x+iy)^{n-1}, 0\right). \end{aligned} \tag{64}$$

Appendix B - The vector potential

Therefore, we have:

$$B_y + iB_x = C_n(x+iy)^{n-1},$$

which is just the multipole field (1).

We have shown that the potential:

$$A_x = 0, \quad A_y = 0, \quad A_z = -B_{\text{ref}} \sum_{n=1}^{\infty} (b_n + ia_n) \frac{\text{Re}(x+iy)^n}{nR_{\text{ref}}^{n-1}}, \quad (65)$$

gives the multipole field (1):

$$B_y + iB_x = B_{\text{ref}} \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x+iy}{R_{\text{ref}}} \right)^{n-1}.$$

Note also that:

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0, \quad (66)$$

so this potential satisfies the Coulomb gauge condition.

Appendix B - The vector potential

Note that the longitudinal field component derived from the multipole potential (65) is:

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = 0. \tag{67}$$

In order to generate a solenoidal field, with $B_z = \text{constant} \neq 0$, we need to introduce non-zero components in A_x, A_y, or both. For example:

$$A_x = -\frac{1}{2} B_{\text{sol}} y, \quad A_y = \frac{1}{2} B_{\text{sol}} x. \tag{68}$$

While it is convenient, for beam dynamics, to work in a gauge with only the z component of the vector potential non-zero, this is not possible for solenoidal fields.

Appendix B - The vector potential

Finally, we give the vector potentials corresponding to 3D fields. In the Cartesian basis (20)-(22):

$$
\begin{aligned}
B_x &= -\tilde{B}(k_x, k_z)\frac{k_x}{k_y} \sin k_x x \, \sinh k_y y \, \sin k_z z, \\
B_y &= \tilde{B}(k_x, k_z) \cos k_x x \, \cosh k_y y \, \sin k_z z, \\
B_z &= \tilde{B}(k_x, k_z)\frac{k_z}{k_y} \cos k_x x \, \sinh k_y y \, \cos k_z z,
\end{aligned}
$$

a possible vector potential is:

$$A_x = 0, \tag{69}$$

$$A_y = \tilde{B}(k_x, k_z)\frac{k_z}{k_x k_y} \sin k_x x \, \sinh k_y y \, \cos k_z z, \tag{70}$$

$$A_z = -\tilde{B}(k_x, k_z)\frac{1}{k_x} \sin k_x x \, \cosh k_y y \, \sin k_z z. \tag{71}$$

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In the Polar basis (36)-(38):

$$B_\rho = \tilde{B}_m(k_z) I'_m(k_z\rho) \sin m\theta \, \cos k_z z,$$

$$B_\theta = \tilde{B}_m(k_z) \frac{m}{k_z\rho} I_m(k_z\rho) \cos m\theta \, \cos k_z z,$$

$$B_z = -\tilde{B}_m(k_z) I_m(k_z\rho) \sin m\theta \, \sin k_z z,$$

a possible vector potential is:

$$A_\rho = -\frac{\rho}{m} I_m(k_z\rho) \cos m\theta \, \sin k_z z, \quad (72)$$

$$A_\theta = 0, \quad (73)$$

$$A_z = -\frac{\rho}{m} I'_m(k_z\rho) \cos m\theta \, \cos k_z z. \quad (74)$$